VECTOR CALCULUS

WITH APPLICATIONS TO PHYSICS

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ILLUSTRATED

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PREFACE.

This volume embodies the lectures given on the subject to *graduate* students over a period of four repetitions. The point of view is the result of many years of consideration of the whole field. The author has examined the various methods that go under the name of VECTOR, and finds that for all purposes of the physicist and for most of those of the geometer, the use of quaternions is by far the simplest in theory and in practice. The various points of view are mentioned in the introduction, and it is hoped that the essential differences are brought out. The tables of comparative notation scattered through the text will assist in following the other methods.

The place of vector work according to the author is in the general field of associative algebra, and every method so far proposed can be easily shown to be an imperfect form of associative algebra. From this standpoint the various discussions as to the fundamental principles may be understood. As far as the mere notations go, there is not much difference save in the actual characters employed. These have assumed a somewhat national character. It is unfortunate that so many exist.

The attempt in this book has been to give a text to the mathematical student on the one hand, in which every physical term beyond mere elementary terms is carefully defined. On the other hand for the physical student there will be found a large collection of examples and exercises which will show him the utility of the mathematical methods. So very little exists in the numerous treatments of the day that does this, and so much that is labeled vector

analysis is merely a kind of short-hand, that it has seemed very desirable to show clearly the actual use of vectors as vectors. It will be rarely the case in the text that any use of the components of vectors will be found. The triplexes in other texts are very seldom much different from the ordinary Cartesian forms, and not worth learning as methods.

The difficulty the author has found with other texts is that after a few very elementary notions, the mathematical student (and we may add the physical student) is suddenly plunged into the profundities of mathematical physics, as if he were familiar with them. This is rarely the case, and the object of this text is to make him familiar with them by easy gradations.

It is not to be expected that the book will be free from errors, and the author will esteem it a favor to have all errors and oversights brought to his attention. He desires to thank specially Dr. C. F. Green, of the University of Illinois, for his careful assistance in reading the proof, and for other useful suggestions. Finally he has gathered his material widely, and is in debt to many authors for it, to all of whom he presents his thanks.

JAMES BYRNIE SHAW.

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VECTOR CALCULUS

CHAPTER I

INTRODUCTION

1. **Vector Calculus.** By this term is meant a system of mathematical thinking which makes use of a special class of symbols and their combinations according to certain given laws, to study the mathematical conclusions resulting from data which depend upon geometric entities called vectors, or physical entities representable by vectors, or more generally entities of any kind which could be represented for the purposes under discussion by vectors. These vectors may be in space of two or three or even four or more dimensions. A geometric vector is a directed segment of a straight line. It has length (including zero) and direction. This is equivalent to saying that it cannot be defined merely by one single numerical value. Any problem of mathematics dependent upon several variables becomes properly a problem in vector calculus. For instance, analytical geometry is a crude kind of vector calculus.

Several systems of vector calculus have been devised, differing in their fundamental notions, their notation, and their laws of combining the symbols. The lack of a uniform notation is deplorable, but there seems little hope of the adoption of any uniform system soon. Existing systems have been rather ardently promoted by mathematicians of the same nationality as their authors, and disagreement exists as to their relative simplicity, their relative directness, and their relative logical exactness. These disagreements arise sometimes merely with regard to the proper manner of representing certain combinations of the symbols, or other matters which are purely matters of convention;

sometimes they are due to different views as to what are the important things to find expressions for; and sometimes they are due to more fundamental divergences of opinion as to the real character of the mathematical ideas underlying any system of this sort. We will indicate these differences and dispose of them in this work.

2. **Bases.** We may classify broadly the various systems of vector calculus as *geometric* and *algebraic*. The former is to be found wherever the desire is to lay emphasis on the spatial character of the entities we are discussing, such as the line, the point, portions of a plane, etc. The latter lays emphasis on the purely algebraic character of the entities with which the calculations are made, these entities being similar to the positive and negative, and the imaginary of ordinary algebra. For the geometric vector systems, the symbolism of the calculus is really nothing more than a short-hand to enable one to follow certain operations upon real geometric elements, with the possibility kept always in mind that these entities and the operations may at any moment be called to the front to take the place of their short-hand representatives. For the algebraic systems, the symbolism has to do with hypernumbers, that is, extensions of the algebraic negative and imaginary numbers, and does not pretend to be the translation of actual operations which can be made visible, any more than an ordinary calculation of algebra could be paralleled by actual geometric or physical operations.

If these distinctions are kept in mind the different points of view become intelligible. The best examples of geometric systems are the *Science of Extension* of Grassmann, with its various later forms, the *Geometry of Dynames* of Study, the *Geometry of Lines* of Saussure, and the *Geometry of Feuillets* of Cailler. The best examples of algebraic systems are the *Quaternions* of Hamilton, *Dyadics* of Gibbs,

Multenions of McAulay, *Biquaternions* of Clifford, *Triquaternions* of Combebiac, *Linear Associative Algebra* of Peirce. Various modifications of these exist, and some mixed systems may be found, which will be noted in the proper places.

The idea of using a calculus of symbols for writing out geometric theorems perhaps originated with Leibniz,[1] though what he had in mind had nothing to do with vector calculus in its modern sense. The first effective algebraic vector calculus was the Quaternions of Hamilton[2] (1843), the first effective geometric vector calculus was the Ausdehnungslehre of Grassmann[3] (1844). They had predecessors worthy of mention and some of these will be noticed.

3. **Hypernumbers.** The real beginning of Vector Calculus was the early attempt to extend the idea of number. The original theory of irrational number was metric,[4] and defined irrationals by means of the segments of straight lines. When to this was added the idea of direction, so that the segments became directed segments, what we now call *vectors,* the numbers defined were not only capable of being irrational, but they also possessed *quality,* and could be negative or positive. Ordinary algebra is thus the first vector calculus. If we consider segments with direction in a plane or in space of three dimensions, then we may call the numbers they define *hypernumbers.* The source of the idea was the attempt to interpret the imaginary which had been created to furnish solutions for any quadratic or cubic. The imaginary appears early in Cardan's work.[5] For instance he gives as solution of the problem of separating 10 into two parts whose product is 40, the values $5 + \sqrt{-15}$, and $5 - \sqrt{-15}$. He considered these numbers as impossible and of no use. Later it was discovered that in the solution of the cubic by Cardan's formula there appeared the sum of two of these impossible

values when the answer actually was real. Bombelli[6] gave as the solution of the cubic $x^3 = 15x + 4$ the form

$$\sqrt[3]{(2 + \sqrt{-121})} + \sqrt[3]{(2 - \sqrt{-121})} = 4.$$

These impossible numbers incited much thought and there came about several attempts to account for them and to interpret them. The underlying question was essentially that of existence, which at that time was usually sought for in concrete cases. The real objection to the negative number was its inapplicability to objects. Its use in a debit and credit account would in this sense give it existence. Likewise the imaginary and the complex number, and later others, needed interpretation, that is, application to physical entities.

4. **Wessel,** a Danish surveyor, in 1797, produced a satisfactory method[7] of defining complex numbers by means of vectors in a plane. This same method was later given by Argand[8] and afterwards by Gauss[9] in connection with various applications. Wessel undertook to go farther and in an analogous manner define hypernumbers by means of directed segments, or vectors, in space of three dimensions. He narrowly missed the invention of quaternions. In 1813 Servois[10] raised the question whether such vectors might not define hypernumbers of the form

$$p \cos \alpha + q \cos \beta + r \cos \gamma$$

and inquired what kind of non-reals p, q, r would be. He did not answer the question, however, and Wessel's paper remained unnoticed for a century.

5. **Hamilton** gave the answer to the question of Servois as the result of a long investigation of the whole problem.[11] He first considered *algebraic couples*, that is to say in our terminology, hypernumbers needing two ordinary numerical

values to define them, and all possible modes of combining them under certain conditions, so as to arrive at a similar couple or hypernumber for the product. He then considered triples and sets of numbers in general. Since -1 and $i = \sqrt{-1}$ are roots of unity, he paid most attention to definitions that would lead to new roots of unity.

His fundamental idea is that the couple of numbers (a, b) where a and b are any positive or negative numbers, rational or irrational, is an entity in itself and is therefore subject to laws of combination just as are single numbers. For instance, we may combine it with the other couple (x, y) in two different ways:

$$(a, b) + (x, y) = (a + x, b + y)$$
$$(a, b) \times (x, y) = (ax - by, ay + bx).$$

In the first case we say we have added the couples, in the second case that we have multiplied them. It is possible to define division also. In both cases if we set the couple on the right hand side equal to (u, v) we find that

$$\partial u/\partial x = \partial v/\partial y, \qquad \partial u/\partial y = - \partial v/\partial x.$$

Pairs of functions u, v which satisfy these partial differential equations Hamilton called *conjugate functions*. The partial differential equations were first given by Cauchy in this connection. The particular couples

$$\epsilon_1 = (1, 0), \qquad \epsilon_2 = (0, 1)$$

play a special role in the development, for, in the first place, any couple may be written in the form

$$(a, b) = a\epsilon_1 + b\epsilon_2$$

and the notation of couples becomes superfluous; in the second place, by defining the products of ϵ_1 and ϵ_2 in various ways we arrive at various algebras of couples. The general

definition would be, using the $\cdot$ for $\times$,

$$\epsilon_1 \cdot \epsilon_1 = c_{111}\epsilon_1 + c_{112}\epsilon_2, \qquad \epsilon_1 \cdot \epsilon_2 = c_{121}\epsilon_1 + c_{122}\epsilon_2,$$
$$\epsilon_2 \cdot \epsilon_1 = c_{211}\epsilon_1 + c_{212}\epsilon_2, \qquad \epsilon_2 \cdot \epsilon_2 = c_{221}\epsilon_1 + c_{222}\epsilon_2.$$

By varying the choice of the arbitrary constants c, and Hamilton considered several different cases, different algebras of couples could be produced. In the case above the c's are all zero except

$$c_{111} = 1, \qquad c_{122} = 1, \qquad c_{212} = 1, \qquad c_{221} = -1.$$

From the character of ϵ_1 it may be regarded as entirely identical with ordinary 1, and it follows therefore that ϵ_2 may be regarded as identical with the $\sqrt{-1}$. On the other hand we may consider ϵ_1 to be a unit vector pointing to the right in the plane of vectors, and ϵ_2 to be a unit vector perpendicular to ϵ_1. We have then a vector calculus practically identical with Wessel's. The great merit of Hamilton's investigation lies of course in its generality. He continued the study of couples by a similar study of triples and then quadruples, arriving thus at Quaternions. His chief difference in point of view from those who followed him and who used the concept of couple, triple, etc. (*Multiple* we will say for the general case), is that he invariably defined *one* product, whereas others define usually several.

6. **Multiples.** There is a considerable tendency in the current literature of vector calculus to use the notion of *multiple*. A vector is usually designated by a triple as (x, y, z), and usually such triple is called a vector. It is generally tacitly understood that the dimensions of the numbers of the triple are the same, and in fact most of the products defined would have no meaning unless this homogeneity of dimension were assumed to hold. We find products defined arbitrarily in several ways. For instance, the *scalar product* of the triples (a, b, c) and (x, y, z)

is $\pm (ax + by + cz)$, the sign depending upon the person giving the definition; the *vector product* of the same two triples is usually given as the triple $(bz - cy, cx - az, ay - bx)$. It is obvious at once that a great defect of such definitions is that the triples involved have no sense until the significance of the first number, the second number, and the third number in each triple is understood. If these depend upon axes for their meaning, then the whole calculus is tied down to such axes, unless, as is usually done, the expressions used in the definitions are so chosen as to be in some respects independent of the particular set of axes chosen. When these expressions are thus chosen as invariants under given transformations of the axes we arrive at certain of the well-known systems of vector analysis. The transformations usually selected to furnish the profitable expressions are the group of orthogonal transformations. For instance, it was shown by Burkhardt[12] that all the invariant expressions or invariant triples are combinations of the three following:

$$\begin{gathered} ax + by + cz, \\ (bz - cy,\ cx - az,\ ay - bx), \\ (al + bm + cn)x + (am - bl)y + (an - cl)z, \\ (bl - am)x + (al + bm + cn)y + (bn - cm)z, \\ (cl - an)x + (cm - bn)u + (al + bm + cn)z. \end{gathered}$$

A study of vector systems from this point of view has been made by Schouten.[13]

7. **Quaternions.** In his first investigations, Hamilton was chiefly concerned with the creation of systems of hypernumbers such that each of the defining units, similar to the ϵ_1 and ϵ_2 above, was a root of unity.[14] That is, the process of multiplication by iteration would bring back the multiplicand. He was actually interested in certain special

cases of abstract groups,[15] and if he had noticed the group property his researches would perhaps have extended into the whole field of abstract groups. In quaternions he found a set of square roots of -1, which he designated by i, j, k, connected with his triples though belonging to a set of quadruples. In his *Lectures on Quaternions*, the first treatise he published on the subject, he chose a geometrical method of exposition, consequently many have been led to think of quaternions as having a geometric origin. However, the original memoirs show that they were reached in a purely algebraic way, and indeed according to Hamilton's philosophy were based on steps of *time* as opposed to geometric steps or vectors.

The geometric definition is quite simple, however, and not so abstract as the purely algebraic definition. According to this idea, numbers have a metric definition, a number, or hypernumber, being the ratio of two vectors. If the vectors have the same direction we arrive at the ordinary numerical scale. If they are opposite we arrive at the negative numbers. If neither in the same direction nor opposite we have a more general kind of number, a hypernumber in fact, which is a quaternion, and of which the ordinary numbers and the negative numbers are merely special cases. If we agree to consider all vectors which are parallel and in the same direction as equivalent, that is, call them *free vectors*, then for every pair of vectors from the origin or any fixed point, there is a quaternion. Among these quaternions relations will exist, which will be one of the objects of study of later chapters.

8. **Möbius** was one of the early inventors of a vector calculus on the geometric basis. In his *Barycentrisches Kalkul*[16] he introduced a method of deriving points from other points by a process called addition, and several

applications were made to geometry. The barycentric calculus is somewhat between a system of homogeneous coordinates and a real vector calculus. His addition was used by Grassmann.

9. **Grassmann** in 1844 published his treatise called *Die lineale Ausdehnungslehre*[17] in which several different processes called multiplication are used for the derivation of geometric entities from other geometric entities. These processes make use of a notation which is practically a sort of short-hand for the geometric processes involved. Grassmann considered these various kinds of multiplication abstractly, leaving out of account the meaning of the elements multiplied. His methods apply to space of N dimensions. In the *symmetric* multiplication it is possible to interchange any two of the factors without affecting the result. In the *circular* multiplication the order may be changed cyclically. In the *lineal* multiplication all the laws hold as well for any factors which are linear combinations of the hypernumbers which define the base, as for those called the base. He studies two species of circular multiplication. If the defining units of the base are $\epsilon_1, \epsilon_2, \epsilon_3 \cdots \epsilon_n$, then we have in the first variety of circular multiplication the laws

$$\epsilon_1{}^2 + \epsilon_2{}^2 + \epsilon_3{}^2 + \cdots + \epsilon_n{}^2 = 0, \qquad \epsilon_i\epsilon_j = \epsilon_j\epsilon_i.$$

In the second variety we have the laws

$$\epsilon_1{}^2 = 0, \qquad \epsilon_2{}^2 = 0, \cdots \epsilon_n{}^2 = 0, \qquad \epsilon_i\epsilon_j = 0, \qquad i \neq j.$$

In the lineal genus of multiplication he studies two species, in the first, called the *algebraic multiplication,* we have the law

$$\epsilon_i\epsilon_j = \epsilon_j\epsilon_i \qquad \text{for all } i, j.$$

while in the second, called the *exterior multiplication,* the interchange of any two factors changes the sign of the

result. Of the latter there are two varieties, the *progressive multiplication* in which the number of dimensions of the geometric figure which is the product is the sum of the dimensions of the factors, while in the other, called *regressive multiplication,* the dimension of the product is the difference between the sum of the dimensions of the factors and N the dimension of the space in which the operation takes place. From the two varieties he deduces another kind called *interior multiplication.*

If we confine our thoughts to space of three dimensions, defined by points, and if ϵ_1, ϵ_2, ϵ_3, ϵ_4 are such points, the progressive exterior product of two, as ϵ_1, ϵ_2, is $\epsilon_1\epsilon_2$ and represents the segment joining them if they do not coincide. The product is zero if they coincide. The product of this into a third point ϵ_3 is $\epsilon_1\epsilon_2\epsilon_3$ and represents the parallelogram with edges $\epsilon_1\epsilon_2$, $\epsilon_1\epsilon_3$ and the other two parallel to these respectively. If all three points are in a straight line the product is zero. The exterior progressive product $\epsilon_1\epsilon_2\epsilon_3\epsilon_4$ represents the parallelepiped with edges $\epsilon_1\epsilon_2$, $\epsilon_1\epsilon_3$, $\epsilon_1\epsilon_4$ and the opposite parallel edges. The regressive exterior product of $\epsilon_1\epsilon_2$ and $\epsilon_1\epsilon_3\epsilon_4$ is their common point ϵ_1. The regressive product of $\epsilon_1\epsilon_2\epsilon_3$ and $\epsilon_1\epsilon_2\epsilon_4$ is their common line $\epsilon_1\epsilon_2$. The complement of ϵ_1 is defined to be $\epsilon_2\epsilon_3\epsilon_4$, and of $\epsilon_1\epsilon_2$ is $\epsilon_3\epsilon_4$, and of $\epsilon_1\epsilon_2\epsilon_3$ is ϵ_4. The interior product of any expression and another is the progressive or regressive product of the first into the complement of the other. For instance, the interior product of ϵ_1 and ϵ_2 is the progressive product of ϵ_1 and $\epsilon_1\epsilon_3\epsilon_4$ which vanishes. The interior product of ϵ_2 and ϵ_2 is the product of ϵ_2 and $\epsilon_1\epsilon_3\epsilon_4$ which is $\epsilon_2\epsilon_1\epsilon_3\epsilon_4$. The interior product of $\epsilon_1\epsilon_2\epsilon_3$ and $\epsilon_1\epsilon_4$ is the product of $\epsilon_1\epsilon_2\epsilon_3$ and $\epsilon_2\epsilon_3$ which would be regressive and be the line $\epsilon_2\epsilon_3$.

We have the same kinds of multiplication if the expressions ϵ are vectors and not points, and they may even be

planes. The interpretation is different, however. It is easy to see that Grassmann's ideas do not lend themselves readily to numerical application, as they are more closely related to the projective transformations of space. In fact, when translated, most of the expressions would be phrased in terms of intersections, points, lines and planes, rather than in terms of distances, angles, areas, etc.

10. **Dyadics** were invented by **Gibbs,**[18] and are of both the algebraic and the geometric character. Gibbs has, like Hamilton, but one kind of multiplication. If we have given two vectors α, β from the same point, their dyad is $\alpha\beta$. This is to be looked upon as a new entity of two dimensions belonging to the point from which the vectors are drawn. It is not a plane though it has two dimensions, but is really a particular and special kind of *dyadic*, an entity of two-dimensional character, such that in every case it can be considered to be the sum of not more than three dyads. Gibbs never laid any stress on the geometric existence of the dyadic, though he stated definitely that it was to be considered as a quantity. His greatest stress, however, was upon the operative character of the dyadic, its various combinations with vectors being easily interpretable. The simplest interpretation is from its use in physics to represent strain.

Gibbs also pushed his vector calculus into space of many dimensions, and into triadic and higher forms, most of which can be used in the theory of the elasticity of crystals. The scalar and vector multiplication he considered as functions of the dyadic, rather than as multiplications, and there are corresponding functions of triadics and higher forms. In this respect his point of view is close to that of Hamilton, the difference being in the use of the dyadic or the quaternion.

11. Other forms of vector calculus can be reduced to

these or to combinations of parts of these. The differences are usually in the notations, or in the basis of exposition.

Notations for One Vector

Greek letters, Hamilton, Tait, Joly, Gibbs.

Italics, Grassmann, Peano, Fehr, Ferraris, Macfarlane. Heun writes $\overline{a}$, $\overline{b}$, $\overline{c}$.

Old English or German letters, Maxwell, Jaumann, Jung, Föppl, Lorentz, Gans, Abraham, Bucherer, Fischer, Sommerfeld.

Clarendon type, Heaviside, Gibbs, Wilson, Jahnke, Timerding, Burali-Forti, Marcolongo.

Length of a vector

T ( ), Hamilton, Tait, Joly.

| |, Gans, Bucherer, Timerding.

Italic corresponding to the vector letter, Wilson, Jaumann, Jung, Fischer, Jahnke. Corresponding small italic, Macfarlane.

Mod. ( ), Peano, Burali-Forti, Marcolongo, Fehr.

Unit of a vector

U ( ), Hamilton, Tait, Joly, Peano.

Clarendon small, Wilson.

$(\ )_1$, Bucherer, Fischer.

Corresponding Greek letter, Macfarlane.

Some write the vector over the length.

Square of a vector

$(\ )^2$. The square is usually positive except in Quaternions, where it is negative.

Reciprocal

$(\ )^{-1}$, Hamilton, Tait, Joly, Jaumann.

$\frac{1}{(\ )}$, Hamilton, Tait, Joly, Fischer, Bucherer.

CHAPTER II

SCALAR FIELDS

1. **Fields.** If we consider a given set of elements in space, we may have for each element one or more quantities determined, which can be properly called *functions of the element.* For instance, at each point in space we may have a temperature, or a pressure, or a density, as of the air. Or for every loop that we may draw in a given space we may have a length, or at some fixed point a potential due to the loop. Again, we may have at each point in space a velocity which has both direction and length, or an electric intensity, or a magnetic intensity. Not to multiply examples unnecessarily, we can see that for a given range of points, or lines, or other geometric elements, we may have a set of quantities, corresponding to the various elements of the range, and therefore constituting a function of the range, and these quantities may consist of numerical values, or of vectors, or of other hypernumbers. When they are of a simple numerical character they are called *scalars,* and the function resulting is a *scalar function.* Examples are the density of a fluid at each point, the density of a distribution of energy, and similar quantities consisting of an amount of some entity per cubic centimeter, or per square centimeter, or per centimeter.

EXAMPLES

(1) *Electricity.* The unit of electricity is the *coulomb,* connected with the absolute units by the equations

$$1\ coulomb = 3\cdot 10^9 \text{ electrostatic units}$$
$$= 10^{-1} \text{ electromagnetic units.}$$

The density of electricity is its amount in a given volume, area, or length divided by the volume, area, or length respectively. The dimensions of electricity will be represented by $[\Theta]$, and for its amount the symbol Θ will be used. For the volume density we will use e, for areal density e', for linear density e''. If the distribution may be considered to be continuous, we may take the limits and find the density at a point.

(2) *Magnetism.* Considering magnetism to be a quantity, we will use for the unit of measurement the *maxwell*, connected with the absolute units by the equation

$$\begin{aligned} 1\ \textit{maxwell} &= 3 \cdot 10^{10} \text{ electrostatic units} \\ &= 1 \text{ electromagnetic unit.} \end{aligned}$$

Sometimes 10^8 maxwells is called a *weber*. The symbol for magnetism will be Φ, the dimensions $[\Phi]$, the densities m, m', m''.

(3) *Action.* This quantity is much used in physics, the principle of least action being one of the most important fundamental bases of modern physics. The dimensions of action are $[\Theta\Phi]$, the symbol we shall use is A, and the unit might be a *quantum,* but for practical purposes a *joule-second* is used. In the case of a moving particle the action at any point depends upon the path by which the particle has reached the point, so that as a function of the points of space it has at each point an infinity of values. A function which has but a single value at a point will be called *monodromic,* but if it has more than one value it will be called *polydromic.* The action is therefore a polydromic function. We not only have action in the motion of particles but we find it as a necessary function of a momentum field, or of an electromagnetic field.

(4) *Energy.* The unit of energy is the *erg* or the *joule*

$= 10^7$ ergs. Its dimensions are $[\Theta\Phi T^{-1}]$, its symbol will be W.

(5) *Activity.* This should not be confused with action. It is measured in *watts*, symbol J, dimensions $[\Theta\Phi T^{-2}]$.

(6) *Energy-density.* The symbol will be U, dimensions $[\Theta\Phi L^{-3} T^{-1}]$.

(7) *Activity-density.* The symbol will be Q, dimensions $[\Theta\Phi L^{-3} T^{-2}]$.

(8) *Mass.* The symbol is M, dimensions $[\Theta\Phi TL^{-2}]$. The unit of mass is the *gram.* A distribution of mass is usually called a distribution of matter.

(9) *Density of mass.* The symbol will be c, dimensions $[\Theta\Phi TL^{-5}]$.

(10) *Potential of electricity.* Symbol V, dimensions $[\Phi T^{-1}]$.

(11) *Potential of magnetism.* Symbol N, dimensions $[\Theta T^{-1}]$.

(12) *Potential of gravity.* Symbol P, dimensions $[\Theta\Phi T^{-1}]$.

2. **Levels.** Points at which the function has the same value, are said to define a *level* surface of the function. It may have one or more sheets. Such surfaces are usually named by the use of the prefixes *iso* and *equi.* For instance, the surfaces in a cloud, which have all points at the same temperature, are called *isothermal surfaces;* surfaces which have points at the same pressure are called *isobaric surfaces;* surfaces of equal density are *isopycnic surfaces;* those of equal specific volume (reciprocal of the density) are the *isosteric surfaces;* those of equal humidity are *isohydric surfaces.* Likewise for gravity, electricity, and magnetism we have *equipotential surfaces.*

3. **Lamellae.** Surfaces are frequently considered for which we have unit difference between the values of the function for the successive surfaces. These surfaces and

the space between them constitute a succession of *unit lamellae.*

If we follow a line from a point A to a point B, the number of unit lamellae traversed will give the difference between the two values of the function at the points A and B. If this is divided by the length of the path we shall have the mean rate of change of the function along the path. If the path is straight and the unit determining the lamellae is made to decrease indefinitely, the limit of this quotient at any point is called the derivative of the function at that point in the given direction. The derivative is approximately the number of unit lamellae traversed in a unit distance, if they are close together.

4. **Geometric Properties.** Monodromic levels cannot intersect each other, though any one may intersect itself. Any one or all of the levels may have nodal lines, conical points, pinch-points, and the other peculiarities of geometric surfaces. These singularities usually depend upon the singularities of the congruence of normals to the surface.

In the case of functions of two variables, the scalar levels will be curves on the surface over which the two variables are defined. Their singularities may be any that can occur in curves on surfaces.

5. **Gradient.** The equation of a level surface is found by setting the function equal to a constant. If, for instance, the point is located by the coordinates x, y, z and the function is $f(x, y, z)$, then the equation of any level is

$$u = f(x, y, z) = C.$$

If we pass to a neighboring point on the same surface we have

$$du = f(x + dx, y + dy, z + dz) - f(x, y, z) = 0.$$

We may usually find functions $\partial f/dx$, $\partial f/d$, $\partial f/dz$,

functions independent of dx, dy, dz, such that

$$du = \partial f/\partial x \cdot dx + \partial f/\partial y \cdot dy + \partial f/\partial z \cdot dz.$$

Now the vector from the first point to the second has as the lengths of its projections on the axes: dx, dy, dz; and if we define a vector whose projections are $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$, which we will call the *Gradient of f*, then the condition $du = 0$ is the condition that the gradient of f shall be perpendicular to the differential on the surface. Hence, if we represent the gradient of f by ∇f, and the differential change from one point to the other by $d\rho$, we see that $d\rho$ is any infinitesimal tangent on the surface and ∇f is along the normal to the surface. It is easy to see that if we differentiate u in a direction not tangent to a level surface of u we shall have

$$du = \partial f/\partial x \cdot dx + \partial f/\partial y \cdot dx + \partial f/\partial z \cdot dz = dC.$$

If the length of the differential path is ds then we shall have*
$du/ds =$ *projection of* ∇f *on the unit vector in the direction of* $d\rho$.
The length of the vector ∇f is sometimes called the gradient rather than the vector itself. Sometimes the negative of the expression used here is called the gradient.

When the three partial derivatives of f vanish for the same point, the intensity of the gradient, measured by its length, is zero, and the direction becomes indeterminate from the first differentials. At such points there are singularities of the function. At points where the function becomes infinite, the gradient becomes indeterminate and such points are also singular points.

6. **Potentials.** The three components of a vector at a point may be the three partial derivatives of the same function as to the coordinates, in which case the vector may be looked upon as the gradient of the integral func-

* Since dx/ds, dy/ds, dz/ds are the direction-cosines of $d\rho$.

tion, which is called a *potential function*, or sometimes a *force function*. For instance, if the components of the velocity satisfy the proper conditions, the velocity is the gradient of a *velocity potential*. These conditions will be discussed later, and the vector will be freed from dependence upon any axes.

7. **Relative Derivatives.** In case there are two scalar functions at a point, we may have use for the concept of the derivative of one with respect to the other. This is defined to be the quotient of the intensity of the gradient of the first by that of the second, multiplied by the cosine of their included angle. If the unit lamellae are constructed, it is easy to see from the definition that the relative derivative of the first as to the second will be the limit of the average or mean of the number of unit sheets of the first traversed from one point to another, along the normal of the second divided by the number of unit sheets of the second traversed at the same time. For instance, if we draw the isobars for a given region of the United States and the simultaneous isotherms, then in passing from a point A to a point B if we traverse 24 isobaric unit sheets and 10 isothermal unit sheets, the average is 2.4 isobars per isotherm.

8. **Unit-Tubes.** If there are two scalar functions in the field, and the unit lamellae are drawn, the unit sheets will usually intersect so as to divide the space under consideration into tubes whose cross-section will be a curvilinear parallelogram. Since the area of such parallelogram is approximately

$$ds_1 ds_2 \csc \theta,$$

where ds_1 is the distance from a unit sheet of the function u to the next unit sheet, and ds_2 the corresponding distance for the function v, while θ is the angle between the surfaces; and since we have, $T\nabla u$ being the intensity of the gradient

of u, and $T\nabla v$ the intensity of the gradient of v,

$$ds_1 = 1/T\nabla u, \qquad ds_2 = 1/T\nabla v$$

the area of the parallelogram will be $1/(T\nabla uT\nabla v \sin \theta)$. Consequently if we count the parallelograms in any plane

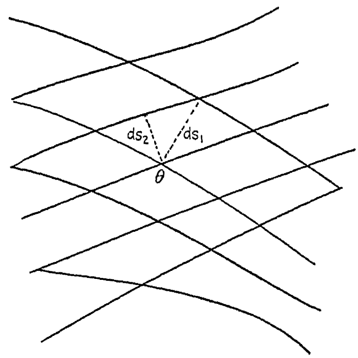

FIG. 1.

cross-section of the two sets of level surfaces, this number is an approximate value of the expression

$$T\nabla uT\nabla v \sin \theta \times \text{area parallelogram}$$

when summed over the plane cross-section. That is to say, the number of these tubes which stand perpendicular to the plane cross-section is the approximate integral of the expression $T\nabla uT\nabla v \sin \theta$ over the area of the cross-section. These tubes are called unit tubes for the same reason that the lamellae are called unit lamellae.

In counting the tubes it must be noticed whether the successive surfaces crossed correspond to an increasing or to a decreasing value of u or of v. It is also clear that when $\sin \theta$ is everywhere 0 the integral must be zero. In such case the three Jacobians

$$\partial(u, v)/\partial(y, z), \qquad \partial(u, v)/\partial(z, x), \qquad \partial(u, v)/\partial(x, y)$$

are each equal zero, and this is the condition that u is a function of v. In case the plane of cross-section is the x, y plane, the first two expressions vanish anyhow, since u, v are functions of x, y only.

It is clear if we take the levels of one of the functions, say u, as the upper and lower parts of the boundary of the cross-section, that in passing from one of the other sides of the boundary along each level of u the number of unit tubes we encounter from that side of the boundary to the opposite side is the excess of the value of v on the second side over that on the first side. If then we count the different tubes in the successive lamellae of u between the two sides of the cross-section we shall have the total excess of those on the second side over those on the first side. That is to say, the number of unit tubes or the integral over the area bounded by level 1 and level 2 of u, and any other two lines which cross these two levels so as to produce a simple area between, is the excess of the sum between the two levels of the values of v on one side over the same sum between the two levels of u on the other side. These graphical solutions are used in Meteorology.

This gives the excess of the integral $\int vdu$ along the second line between the two levels of u, over the same integral along the first line. It represents the increase of this integral in a change of path from one line to the other. For instance if the integral is energy, the number of tubes is the amount of energy stored or released in the passage from one line to the other, as in a cyclone. The number of tubes for any closed path is the approximate integral $\int vdu$ around the path.

EXERCISES.

1. If the density varies as the distance from a given axis, what are the isopycnic surfaces?

2. A rotating fluid mass is in equilibrium under the force of gravity, the hydrostatic pressure, and the centrifugal force. What are the levels? Show that the field of force is conservative.

3. The isobaric surfaces are parallel planes, and the isopycnic surfaces are parallel planes at an angle of 10° with the isobaric planes. What is the rate of change of pressure per unit rate of change of density along a line at 45° with the isobaric planes?

4. If the pressure can be stated as a function of the density, what conditions are necessary? Are they sufficient? What is the interpretation with regard to the levels?

5. Three scalar functions have a functional relation if their Jacobian vanishes. What does this mean with regard to their respective levels?

6. If the isothermal surfaces are spheres with center at the earth's center, the temperature sheets for decrease of one degree being 166.66 feet apart, and if the isobaric levels are similar spheres, the pressure being given by

$$\log B = \log B_0 - 0.0000177(z - z_0),$$

where B_0 is the pressure at z_0 feet above the surface of the earth, what is the relative derivative of the temperature as to the pressure, and the pressure as to the temperature?

7. To find the maximum of $u(x, y, z)$ we set $du = 0$. If there is also a condition to be fulfilled, $v(x, y, z) = 0$, then $dv = 0$ also.

These two equations in dx, dy, dz must be satisfied for all compatible values of dx, dy, dz, and we must therefore have

$$\frac{\partial u}{\partial x} : \frac{\partial u}{\partial y} : \frac{\partial u}{\partial z} : = \frac{\partial v}{\partial x} : \frac{\partial v}{\partial y} : \frac{\partial v}{\partial z},$$

which is equivalent to the single vector equation

$$\nabla u = w \nabla v.$$

What does this mean in terms of the levels? The unit tubes?

If there is also another equation of condition $t(x, y, z) = 0$ then also $dt = 0$ and the Jacobian of the three functions u, v, t must equal zero. Interpret.

8. On the line of intersection of two levels of two different functions the values of both functions remain constant. If we differentiate a third function along the locus in question, the differential vanishing everywhere, what is the significance?

9. If a field of force has a potential, then a fluid, subject to the force and such that its pressure is a function of the density and the temperature, will have the equipotential levels for isobaric levels also. The density will be the derivative of the pressure relative to the potential. Show therefore that equilibrium is not possible unless the isothermals are also the levels of force and of pressure.

[$p = p(c, T)$, and $\nabla p = c\nabla v = p_c\nabla c + p_T\nabla T$.

If then $\nabla c = 0$, $c\nabla v = p_T\nabla T$.]

10. If the full lines below represent the profiles of isobaric sheets, and the dotted lines the profiles of isosteric sheets, count the unit tubes between the two verticals, and explain what the number means. If they were equipotentials of gravity and isopycnic surfaces, what would the number of unit tubes mean?

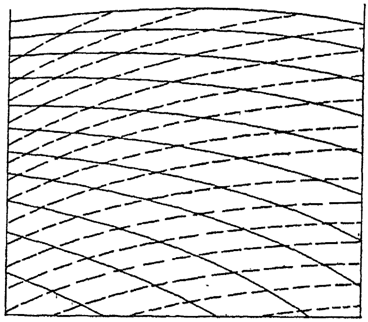

FIG. 2.

11. If $u = y - 12x^3$ and $v = y + x^2 + \frac{1}{4}x$, find ∇u and ∇v and $T\nabla uT\nabla v\cdot\sin\theta$, and integrate the latter over the area between $x = 0$, $x = 1$, $y = 0$, $y = 12$. Draw the lines.

12. If $u = ax + by + cz$ and $v = x^2 + y^2 + z^2$, find ∇u and ∇v and $T\nabla uT\nabla v\cdot\sin\theta$ and integrate the latter expression over the surface of a cylinder whose axis is in the direction of the z axis. Find the derivative of each relative to the other.

CHAPTER III

VECTOR FIELDS

1. **Hypercomplex Quantity.** In the measurement of quantity the first and most natural invention of the mind was the ordinary system of integers. Following this came the invention of fractions, then of irrational numbers. With these the necessary list of numbers for mere measurement of similar quantities is closed, up to the present time. Whether it will be necessary to invent a further extension of number along this line remains for the future to show.

In the attempt to solve equations involving ordinary numbers, it became necessary to invent negative numbers and imaginary numbers. These were known and used as fictitious numbers before it was noticed that quantities also are of a negative or an "imaginary" character. We find instances everywhere. In debit and credit, for example, we have quantity which may be looked upon as of two different kinds, like iron and time, but the most logical conception is to classify debits and credits together in the single class *balance*. One's balance is what he is worth when the debits and credits have been compared. If the preponderance is on the side of debit we consider the balance negative, if on the side of credit we consider the balance positive. Likewise, we may consider motion in each direction of the compass as in a class by itself, never using any conception of measurement save the purely numerical one of comparing things which are exactly of the same kind together. But it is more logical, and certainly more general, to consider motions in all directions of the compass and of any distances as all belonging to a single class of quantity.

In that case the comparison of the different motions leads us to the notion of complex numbers. When Wessel made his study of the vectors in a plane he was studying the hypernumbers we usually call "the complex field." The hypernumbers had been studied in themselves before, but were looked upon (rightly) as being creations of the mind and (in that sense correctly) as having no existence in what might be called the real world. However, their deduction from the vectors in a plane showed that they were present as relations of quantities which could be considered as alike. Again when Steinmetz made use of them in the study of the relations of alternating currents and electromotive forces, it became evident that the so-called power current and wattless current could be regarded as parts of a single complex current, and similarly for the electromotive forces. The laws of Ohm and Kirchoff could then be generalized so as to be true for the new complex quantities. In this brief history we find an example of the interaction of the developments of mathematics. The inventions of mathematics find instances in natural phenomena, and in some cases furnish new conceptions by which natural phenomena can be regarded as containing elements that would ordinarily be completely overlooked.

In space of three (or more) dimensions, the vectors issuing from a point in all directions and of all lengths furnish quantities which may be considered to be all of the same kind, on one basis of classification. Therefore, they will define certain ratios or relations which may be called hypernumbers. This is the class of hypernumbers we are particularly concerned with, though we shall occasionally notice others. Further, any kind of quantity which can be represented completely for certain purposes by vectors issuing from a point we will call vector quantity.

Such quantities, for instance, are motions, velocities, accelerations, at least in the Newtonian mechanics, forces, momenta, and many others. The object of *VECTOR CALCULUS* is to study these hypernumbers in relation to their corresponding quantities, and to derive an algebra capable of handling them.

We do not consider a vector as a mere triplex of ordinary numbers. Indeed, we shall consider two vectors to be identical when they represent or can represent the same quantity, even though one is expressed by a certain triplex, as ordinary Cartesian coordinates, and the other by another triplex, as polar coordinates. The numerical method of defining the vector will be considered as incidental.

2. **Notation.** We shall represent vectors for the most part by Greek small letters. Occasionally, however, as in Electricity, it will be more convenient to use the standard symbols, which are generally Gothic type. As indicated on page 12 there is a great variety of notation, and only one principle seems to be used by most writers, namely that of using heavy type for vectors, whatever the style of type. In case the vector is from the origin to the point (x, y, z) it may be indicated by

$$\rho_{x,\,y,\,z},$$

while for the same point given by polar coordinates r, φ, θ we may use

$$\rho_{r,\,\varphi,\,\theta},$$

In case a vector is given by its components as X, Y, Z we will indicate it by

$$\xi_{X,\,Y,\,Z}$$

3. **Equivalence.** All vectors which have the same direction and same length will be considered to be equivalent. Such vectors are sometimes called *free vectors.* The term vector will be used throughout this book, however, with no other meaning.

In case vectors are equivalent only when they lie on the same line, and have the same direction and length, they will be called *glissants*. A force applied to a rigid body must be considered to be a glissant, not a vector. In case vectors are equivalent only when they start at the same point and coincide, they will be called *radials*. The resultant moment of a system of glissants with respect to a point A is a radial from A.

The equivalence of two vectors

$$\alpha = \beta$$

implies the existence of equalities infinite in number, for their projections on any other lines will then be equal. The infinite set of equalities, however, is reducible in an infinity of ways to three independent equalities. For instance, we may write either

$\alpha_x = \beta_x,\ \alpha_y = \beta_y,\ \alpha_z = \beta_z,$ or $\alpha_r = \beta_r,\ \alpha_\varphi = \beta_\varphi, \alpha_\varphi = \beta_\varphi.$

The equivalence of two glissants implies sets of equalities reducible in every case to five independent equalities. The equivalence of two radials reduces to sets of six equalities.

4. **Vector Fields.** Closely allied to the notion of radial is that of vector field. A vector field is a system of vectors each associated with a point of space, or a point of a surface, or a point of a line or curve. The vector is a function of the position of the point which is itself usually given by a vector, as ρ. The vector function may be monodromic or polydromic. We will consider some of the usual vector fields.

EXAMPLES

(1) **Radius Vector.** ρ $[L]$. This will usually be indicated by ρ. In case it is a function of a single parameter, as t, the points defined will lie on a curve;* in case it is a function

* We are discussing mainly ordinary functions, not the "pathologic type."

of two parameters, u, v, the points defined will lie on a surface. The term vector was first introduced by Hamilton in this sense. When we say that the field is ρ, we mean that at the point whose vector is ρ measured from the fixed origin, there is a field of velocity, or force, or other quantity, whose value at the point is ρ.

(2) **Velocity.** σ $[LT^{-1}]$. Usually we will designate velocity by σ. In the case of a moving gas or cloud, each particle has at each point of its path a definite velocity, so that we can describe the entire configuration of the moving mass at any instant by stating what function σ is of ρ, that is, for the point at the end of the radius vector ρ assign the velocity vector. The path of a moving particle will be called a trajectory. At each point of the path the velocity σ is a tangent of the trajectory.

If we lay off from a fixed point the vectors σ which correspond to a given trajectory, their terminal points will lie on a locus called by Hamilton the *hodograph* of the trajectory. For instance, the hodographs of the orbits of the planets are circles, to a first approximation. If we multiply σ by dt, which gives it the dimensions of length, namely an infinitesimal length along the tangent of the trajectory, the differential equation of the trajectory becomes

$$d\rho = \sigma dt.$$

The integral of this in terms of t gives the equation of the trajectory.

(3) **Acceleration.** τ $[LT^{-2}]$. An acceleration field is similar to a velocity field except in dimensions. The acceleration is the rate of change of the vector velocity at a point, consequently, if a point describes the hodograph of a trajectory so that its radius vector at a given time is the velocity in the trajectory at that time, the acceleration will be a

tangent to the hodograph, and its length will be the velocity of the moving point in the hodograph. We will use τ to indicate acceleration.

(4) **Momentum Density.** Γ $[\Phi\Theta L^{-4}]$. This is a vector function of points in space and of some number which can be attached to the point, called density. In the case of a moving cloud, for instance, each point of the cloud will have a velocity and a density. The product of these two factors will be a vector whose direction is that of the velocity and whose length is the product of the length of the velocity vector and the density. However, momentum density may exist without matter and without motion. In electrodynamic fields, such as could exist in the very simple case of a single point charge of electricity and a single magnet pole at a point, we also have at every point of space a momentum density vector. This may be ascribed to the hypothetical motion of a hypothetical ether, but the essential feature is the existence of the field. If we calculate the integral of the projection of the momentum density on the tangent to a given curve from a point A to a point B, the value of the integral is the *action* of an infinitesimal volume, an action density, along that path from A to B. The integration over a given volume would give the total action for all the particles over their various paths. This would be a minimum for the paths actually described as compared with possible paths. *Specific momentum* is momentum density of a moving mass.

(5) **Momentum.** Y $[\Phi\Theta L^{-1}]$. The volume integral of momentum density or specific momentum is momentum. Action is the line-integral of momentum.

(6) **Force Density.** F $[\Phi\Theta L^{-4}T^{-1}]$. If a field of momentum density is varying in time then at each point there is a vector which may be called force-density, the time derivative

of the momentum density. Such cases occur in fields due to moving electrons or in the action of a field of electric intensity upon electric density, or magnetic intensity on magnetic density.

(7) **Force.** $\mathbf{X}$ $[\Phi\Theta L^{-1}T^{-1}]$. The unit of force has received a name, *dyne*. It is the volume integral of force density. The time integral of a field of force is momentum. In a stationary field of force the line integral of the field for a given path is the difference in energy between the points at the ends of the path, or what is commonly called *work*. In case the field is conservative the integral has the same value for all paths (which at least avoid certain singular points), and depends only on the end points, This takes place when the field is a gradient field of a force-function, or a potential function. If we project the force upon the velocity at each point where both fields exist, the time integral of the scalar quantity which is the product of the intensity of the force, the intensity of the velocity and the cosine of the angle between them, is the activity at the point.

(8) **Flux Density.** Ω $[L^2T^{-1}]$. In the case of the flow of an entity through a surface the limiting value of the amount that flows normally across an infinitesimal area is a vector whose direction is that of the outward normal of the surface, and whose intensity is the limit. In the case of a flow not normal to the surface across which the flux is to be determined, we nevertheless define the flux density as above. The *flux* across any surface becomes then the surface integral of the projection of the flux density on the normal of the surface across which the flux is to be measured.

Flux density is an example of a vector which depends upon an area, and is sometimes called a *bivector*. The notion of two vectors involved in the term bivector may

be avoided by the term *cycle,* or the term *feuille.* It is also called an *axial vector,* in opposition to the ordinary vectors, called *polar vectors.* The term axial is applicable in the sense that it is the axis or normal of a portion of a surface. The portion (feuille, cycle) of the surface is traversed in the positive direction in going around its boundary, that is, with the surface on the left-hand. If the direction of the axial vector is reversed, we also traverse the area attached in the reverse direction, so that in this sense the axial vector may be regarded as invariant for such change while the polar vector would not be invariant. The distinction is not of much importance. The important idea is that of *areal integration* for the flux density or any other so-called axial vector, while the polar vector is subject only to *linear integration.* We meet the distinction in the difference below between the induction vectors and the intensity vectors.

(9) **Energy Density Current.** $\mathbf{R}$ $[\Phi\Theta L^{-2}T^{-2}]$. When an energy density has the idea of velocity attached to it, it becomes a vector with the given dimensions. In such case we consider it as of the nature of a flux density.

(10) **Energy Current.** Σ $[\Phi\Theta T^{-2}]$. If a vector of energy density current is multiplied by an area we arrive at an energy current.

(11) **Electric Density Current.** $\mathbf{J}$ $[\Theta L^{-2}T^{-1}]$. A number of moving electrons will determine an average density per square centimeter across the line of flow, and the product of this into a velocity will give an electric density current. To this must also be added the time rate of change of electric induction, which is of the same dimensions, and counts as an electric density current.

(12) **Electric Current.** $\mathbf{C}$ $[\Theta T^{-1}]$. The unit is the *ampere* $= 3\cdot 10^9$ *e.s. units* $= 10^{-1}$ *e.m. units.* This is the product of an electric density current by an area.

(13) **Magnetic Density Current.** $\mathbf{G}$ $[\Phi L^{-2} T^{-1}]$. Though there is usually no meaning to a moving mass of magnetism, nevertheless, the time rate of change of magnetic induction must be considered to be a current, similar to electric current density.

(14) **Magnetic Current.** $\mathbf{K}$ $[\Phi T^{-1}]$. The unit is the *heavyside* = 1 *e.m. unit* = $3 \cdot 10^{10}$ *e.s. units*. In the phenomena of magnetic leakage we have a real example of what may be called magnetic current.

Both electric current and magnetic current may also be scalars. For instance, if the corresponding flux densities are integrated over a given surface the resulting scalar values would give the rate at which the electricity or the magnetism is passing through the surface per second. In such case the symbols should be changed to corresponding Roman capitals.

(15) **Electric Intensity.** $\mathbf{E}$ $[\Phi L^{-1} T^{-1}]$. When an electric charge is present in any portion of space, there is at each point of space a vector of a field called the field of electric intensity. The same situation happens when lines of magnetic induction are moving through space with a given velocity. The electric intensity will be perpendicular to both the line of magnetic induction and to the velocity it has, and equal to the product of their intensities by the sine of their angle.

The electric intensity is of the nature of a polar vector and its flux, or surface integral over any surface has no meaning. Its line integral along any given path, however, is called the difference of *voltage* between the two points at the ends of the path, for that given path. The unit of voltage is the *volt* = $\frac{1}{3} \cdot 10^{-2}$ *e.s. units* = 10^8 *e.m. units*. The symbol for voltage is V $[\Phi T^{-1}]$. Its dimensions are the same as for scalar electric potential, or magnetic current.

(16) **Electric Induction.** $\mathbf{D}$ $[\Theta L^{-2}]$. The unit is the *line* $= 3 \cdot 10^9$ *e.s. units* $= 10^{-1}$ *e.m. units*. This vector usually has the same direction as electric intensity, but in non-isotropic media, such as crystals, the directions do not agree. It is a linear function of the intensity, however, ordinarily indicated by

$$\mathbf{D} = \kappa(\mathbf{E})$$

where κ is the symbol for a linear operator which converts vectors into vectors, called here the *permittivity*, $[\Theta\Phi^{-1}L^{-1}T]$, measurable in farads per centimeter. In isotropic media κ is a mere numerical multiplier with the proper dimensions, which are essential to the formulae, and should not be neglected even when $\kappa = 1$. The flux is measured in coulombs.

(17) **Magnetic Intensity.** $\mathbf{H}$ $[\Theta L^{-1}T^{-1}]$. The field due to the poles of permanent magnets, or to a direct current traversing a wire, is a field of magnetic intensity. In case we have moving lines of electric induction, there is a field of magnetic intensity. It is of a polar character, and its flux through a surface has no meaning. The line integral between two points, however, is called the *gilbertage* between the points along the given path, the unit being the *gilbert* $= 1$ *e.m. unit* $= 3 \cdot 10^{10}$ *e.s. units*. The symbol is N $[\Theta T^{-1}]$. Its dimensions are the same as those of scalar magnetic potential, or electric current.

(18) **Magnetic Induction.** $\mathbf{B}$ $[\Phi L^{-2}]$. The unit is the *gauss* $= 1$ *e.m. unit* $= 3 \cdot 10^{10}$ *e.s. units*. The direction is usually the same as that of the intensity, but in any case is given by a linear vector operator so that we have

$$\mathbf{B} = \mu(\mathbf{H})$$

where μ is the *inductivity*, $[\Phi\Theta^{-1}L^{-1}T]$, measurable in henrys per centimeter. The flux is measured in maxwells.

(19) **Vector Potential of Electric Induction.** Υ $[\Theta L^{-1}]$. A vector field may be related to another vector field in a certain manner to be described later, such that the first can be called the *vector potential* of the other.

(20) **Vector Potential of Magnetic Induction.** Ψ $[\Phi L^{-1}]$. This is derivable from a field of magnetic induction. This and the preceding are line-integrable.

(21) **Hertzian Vectors.** Θ, Φ. These are line integrals of the preceding two, and are of a vector nature.

5. **Vector Lines.** If we start at a given point of a vector field and consider the vector of the field at that point to be the tangent to a curve passing through the point, the field will determine a set of curves called a *congruence*, since there will be a two-fold infinity of curves, which will at every point have the vector of the field as tangent. If the field is represented by σ, a function of ρ, the vector to a point of the field, then the differential equation of these lines of the congruence will be

$$d\rho = \sigma dt,$$

where dt is a differential parameter. From this we can determine the equation of the lines of the congruence, involving an arbitrary vector, which, however, will not have more than two essential constants. For instance, if the field is given by $\sigma = \rho$, then $d\rho = \rho dt$, and $\rho = \alpha e^t$, where α is a constant unit vector. The lines are, in this case, the rays emanating from the origin.

The lines can be constructed approximately by starting at any given point, thence following the vector of the field for a small distance, from the point so reached following the new vector of the field a small distance, and so proceeding as far as necessary. This will trace approximately a vector line. Usually the curves are unique, for if the field is monodromic at all points, or at points in general, the

curves must be uniquely determined as there will be at any point but one direction to follow. Two vector lines may evidently be tangent at some point, but in a monodromic field they cannot intersect, except at points where the intensity of the field is zero, for vectors of zero intensity are of indeterminate direction. Such points of intersection are *singular points* of the field, and their study is of high importance, not only mathematically but for applications. In the example above the origin is evidently a singular point, for at the origin $\sigma = 0$, and its direction is indeterminate.

6. **Vector Surfaces, Vector Tubes.** In the vector field we may select a set of points that lie upon a given curve and from each point draw the vector line. All such vector lines will lie upon a surface called a *vector surface,* which in case the given curve is closed, forming a loop, is further particularized as a *vector tube.* It is evident that the vector lines are the characteristics of the differential equation $d\rho = \sigma dt$, which in rectangular coordinates would be equivalent to the equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}.$$

In case these equations are combined so as to give a single exact equation, the integral will (since it must contain a single arbitrary constant) be the equation of a family of vector surfaces. The vector lines are the intersections of two such families of vector surfaces. The two families may be chosen of course in infinitely many different ways. Usually, however, as in Meteorology, those surfaces are chosen which have some significance. When a vector tube becomes infinitesimal its limit is a vector line.

7. **Isogons.** If we locate the points at which σ has the

same direction, they determine a locus called an isogon for the field. For instance, we might locate on a weather map all the points which have the same direction of the wind. If isogons are constructed in any way it becomes a simple matter to draw the vector lines of the field. Machines for the use of meteorologists intended to mark the isogons have been invented and are in use.* As an instance consider the vector field

$$\sigma = (2x,\ 2y,\ -z).$$

An isogon with the points at which σ has the direction whose cosines are l, m, n is given by the equations

$$2x : 2y : -z = l : m : n$$

or

$$2x = lt, \qquad 2y = mt, \qquad z = -nt.$$

It follows that the vector to any point of this isogon is given by

$$\rho = t(l,\ m,\ n) - (0,\ 0,\ 3nt).$$

That is to say, to draw the vector ρ to any point of the isogon we draw a ray from the origin in the direction given, then from its outer end draw a parallel to the Z direction backward three times the length of the Z projection of the segment of the ray. The points so determined will evidently lie on straight lines in the same plane as the ray and its projection on the XY plane, with a negative slope twice the positive slope of the ray. The tangents of the vector lines passing through the points of the isogon will then be parallel to the ray itself. The vector lines are drawn approximately by drawing short segments along the isogon parallel to its corresponding ray, and selecting points such that these short segments will make continuous lines in

* Sandström: Annalen der Hydrographie und Maritimen Meteorologie (1909), no. 6, pp. 242 et. seq. Bjerknes: Dynamic Meteorology. See plates, p. 50.

passing to adjacent isogons. The figure illustrates the method. All the vector lines are found by rotating the figure about the X axis 180°, and then rotating the figure so produced about the Z axis through all angles.

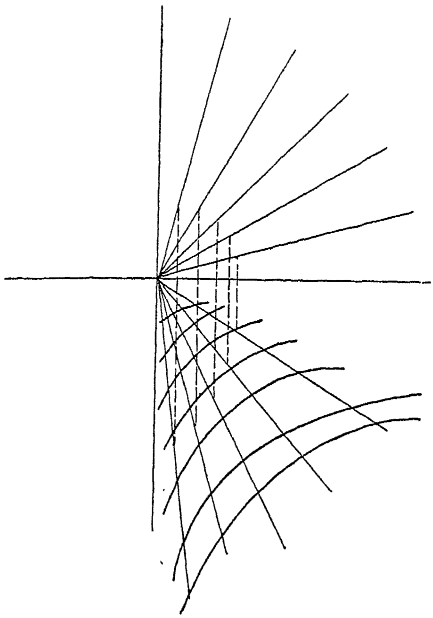

FIG. 3.

8. **Singularities.** It is evident in the example preceding that there are in the figure two lines which are different from the other vector lines, namely, the Z axis and the line which is in the XY plane. Corresponding to the latter would be an infinity of lines in the XY plane passing through the origin. These lines are peculiar in that the other vector lines are asymptotic to them, while they are themselves vector lines of the field. A method of studying the vector lines in the entire extent of the plane in which they lie was used by Poincaré. It consists in placing a sphere tangent

to the plane at the origin. Lines are then drawn from the center of the sphere to every point of the plane, thus giving two points on the sphere, one on the hemisphere next the plane and one diametrically opposite on the hemisphere away from the plane. The points at infinity in the plane correspond to the equator or great circle parallel to the plane. In this representation every algebraic curve in the plane gives a closed curve or *cycle* on the sphere. In the present case, the axes in the plane give two perpendicular great circles on the sphere, and the vector lines will be loops tangent to these great circles at points where they cross the equator. These loops will form in the four lunes of the sphere a system of closed curves which Poincaré calls a topographical system. The equator evidently belongs to the system, being the limit of the loops as they grow narrower. The two great circles corresponding to the axes also belong to the system, being the limits of the loops as they grow larger. If a point describes a vector line its projection on the sphere will describe a loop, and could never leave the lune in which the projection is situated. The points of tangency are called *nodes*; the points which represent the origin, and through which only the singular vector lines pass, are called *fauces*.

9. **Singular Points.** The simplest singular lines depend upon the singular points and these are found comparatively simply. The singular points occur where

$$\sigma = 0 \qquad \text{or} \qquad \sigma = \infty.$$

Since we may multiply the components of σ by any expressions and still have the lines of the field the same, we may equally suppose that the components of σ are reduced to as low terms as possible by the exclusion of common factors of all of them. We will consider first the singular

points for fields in space, then those cases which have lines every point of which is a singular point, which will include the cases of plane fields, since these latter may be considered to represent the fields produced by moving the plane field parallel to itself. The classification given by Poincaré is as follows.

(1) *Node.* At a node there may be many directions in which vector lines leave the point. An example is $\sigma = \rho$. At the origin, it is easy to see, $\sigma = 0$, and it is not possible to start at the origin and follow any definite direction. In fact the vector lines are evidently the rays from the origin in all directions. There is no other singular point at a finite distance. If, however, we consider all the rays in any one plane, and for this plane construct the sphere of projection, we see that the lines correspond to great circles on the sphere which all pass through the origin and the point diametrically opposite to it. This ideal point may be considered to be another node, so that all the vector lines run from node to node, in this case. Every vector line which does not terminate in a node is a spiral or a cycle.

(2) *Faux.* From a faux* there runs an infinity of vector lines which are all on one surface, and a single isolated vector line which intersects the surface at the faux. The surface is a singular surface since every vector line in it through the faux is a singular line. The singular surface is approached asymptotically by all the vector lines not singular.

An example is given by

$$\sigma = (x, y, -z).$$

The vector lines are to be found by drawing all equilateral hyperbolas in the four quadrants of the ZX plane, and then

* Poincaré uses the term *col*, meaning mountain pass, for which faux is Latin.

rotating this set of lines about the Z axis. Evidently all rays in the XY plane from the origin are singular lines, as well as the Z axis. Where fauces occur the singular lines through them are asymptotes for the nonsingular lines. If

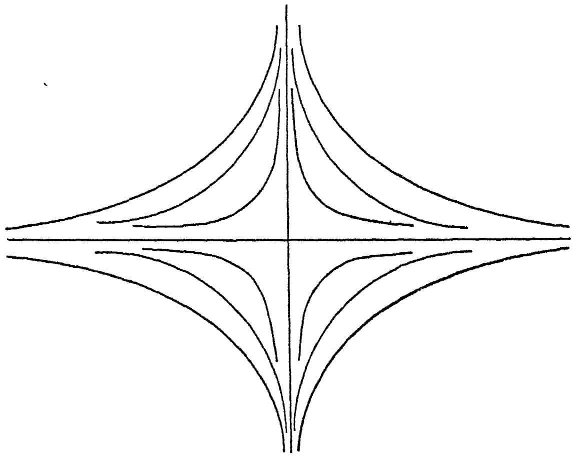

FIG. 4.

we consider any plane through the Z axis, the system of equilateral hyperbolas will project onto its sphere as cycles tangent on the equator to the great circles which represent the singular lines in that plane. From this point of view we really should consider the two rays of the Z axis as separate from each other, so that the upper part of the Z axis and the singular ray perpendicular to it, running in the same general direction as the other vector lines, would constitute a vector line with a discontinuity of direction, or with an angle. Such a vector line to which the others are tangent at points at infinity only is a boundary line in the sense that on one side we have infinitely many vector lines which form cycles (in the sense defined) while on the other sides we have vector lines which belong to different systems of cycles.

A simple case of this example might arise in the inward flow of air over a level plane, with an ascending motion which increased as the air approached a given vertical line, becoming asymptotic to this vertical line. In fact, a small fire in the center of a circular tent open at the bottom for a small distance and at the vertex, would give a motion to the smoke closely approximating to that described.

A singular line from a faux runs to a node or else is a spiral or part of a cycle which returns to the faux.

An example that shows both preceding types is the field

$$\sigma = (x^2 + y^2 - 1,\ 5xy - 5,\ mz).$$

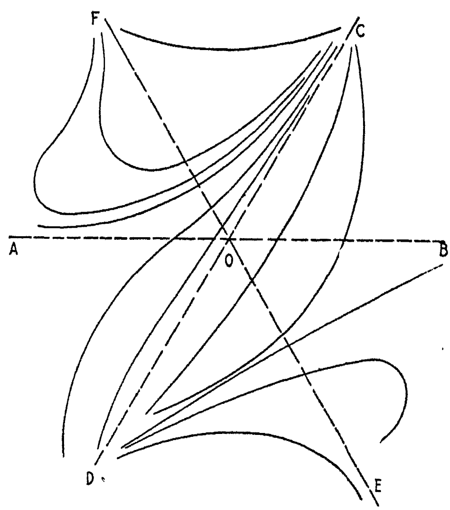

FIG. 5.

In the XY plane the singular points are at infinity as follows: A at the negative end of the X axis, and B at the positive end, both fauces; C at the end of the ray whose direction is $\tan^{-1} 2$, in the first quadrant, D at the end of the ray of direction $\tan^{-1} 2$ in the third quadrant; E at the end of the

ray of direction $\tan^{-1} - 2$ in the fourth quadrant; and F at the end of the ray of $\tan^{-1} - 2$ in the second quadrant, these four being nodes. Vector lines run from E to D separated from the rest of the plane by an asymptotic division line from B to D; from C to D on the other side of this division line, separated from the third portion of the plane by an asymptotic division line from C to A; and from C to F in the third portion of the plane. The figure shows the typical lines of the field.

(3) *Focus.* At a focus the vector lines wind in asymptotically, either like spirals wound towards the vertex of a spindle produced by rotating a curve about one of its tangents, one vector line passing through the focus, or they are like spirals wound around a cone towards the

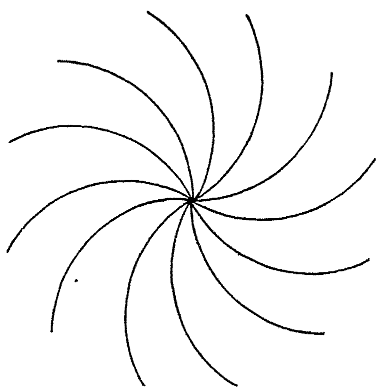

FIG. 6.

vertex. As an example

$$\sigma = (x + y, y - x, z).$$

The Z axis is a single singular line through the origin, which is a singular point, a focus in this case. The XY plane contains vector lines which are logarithmic spirals wound in towards the origin. The other vector lines are spirals

wound on cones of revolution, their projections on XY being the logarithmic spirals. By changing z to az we would have different surfaces depending upon whether

$$a < 1 \qquad \text{or} \qquad 1 < a.$$

In case a spiral winds in onto a cycle, the successive turns approaching the cycle asymptotically, the cycle is called a *limit cycle*. In this example the line at infinity in the XY plane, or the corresponding equator on its sphere, is a limit cycle. It is clear that the spirals on the cones wind outward also towards the lines at infinity as limit cycles. From this example it is plain that vector lines which are spiral may start asymptotically from a focus and be bounded by a limit cycle. The limit cycle thus divides the plane or the surface upon which they lie into two mutually exclusive regions. Vector lines may also start from a limit cycle and proceed to another limit cycle.

As an example of vector lines of both kinds consider the field

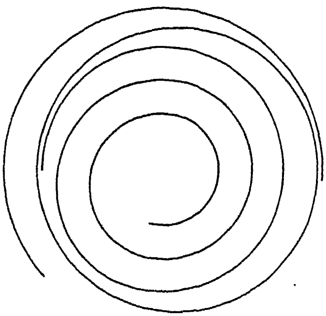

FIG. 7.

$$\sigma = (r^2 - 1,\ r^2 + 1,\ mz),$$

where the first component is in the direction of a ray in the XY plane from the origin, the second perpendicular to

this in the XY plane, and the third is parallel to the Z axis. The vector lines in the singular plane, the XY plane, are spirals with the origin as a focus for one set, which wind around the focus negatively and have the unit circle as a limit cycle, while another set wind around the unit circle in the opposite direction, having the line at infinity as a limit cycle. The polar equation of the first set is $r^{-1} - r = e^{\theta+c}$, of the second set is $r - r^{-1} = e^{\theta+c}$.

An example with all the preceding kinds of singularities is the field

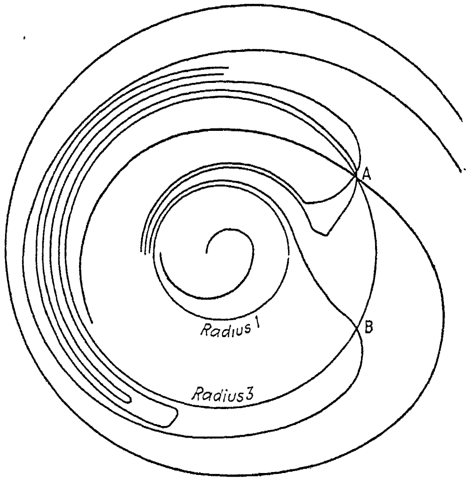

FIG. 8.

$$\sigma = ([r^2 - 1)(r^2 - 9)], (r^2 - 2r \cos \theta - 8), mz)$$

with directions for the components as in the preceding example. The singular points are the origin, a focus; the point A ($r = 3$, $\theta = + \cos^{-1} \frac{1}{6}$), a node; the point B ($r = 3$, $\theta = - \cos^{-1} \frac{1}{6}$), a faux. The line at infinity is a limit cycle, as well as the circle $r = 1$, which is also a vector line. The circle $r = 3$ is a vector line which is a cycle,

starting at the faux, passing through the node and returning to the faux. The vector lines are of three types, the first being spirals that wind asymptotically around the focus, out to the unit circle as limit cycle; the second start at the node A and wind in on the unit circle as limit cycle; the third start at the node A and wind out to the line at infinity as unit cycle. The second set dip down towards the faux. The exceptional vector lines are the line at infinity, the unit circle, both being limit cycles; the circle of radius 3; a vector line which on the one side starts at the faux B winding in on the unit circle, and on the other side starts at the faux B winding outward to the line at infinity as limit cycle. The last two are asymptotic division lines of the regions. The figure exhibits the typical curves.

(4) *Faux-Focus.* This type of singular point has passing through it a singular surface which contains an infinity of spirals having the point as focus, while an isolated vector line passes through the point and the surface. No other surfaces through the vector lines approach the point. An instance is the field

$$\sigma = (x, y, -z).$$

The Z axis is the isolated singular line, while the XY plane is the singular plane. In it there is an infinity of spirals with the origin as focus and the line at infinity as limit cycle. All other vector lines lie on the surfaces $rz = const.$ These do not approach the origin.

(5) *Center.* At a center there is a vector line passing through the singular point, and not passing through this singular line there is a singular surface, with a set of loops or cycles surrounding the center, and shrinking upon it. There is also a set of surfaces surrounding the isolated singular line like a set of sheaths, on each of which there are vector lines winding around helically on it with a decreasing

pitch as they approach the singular surface, which they therefore approach asymptotically. As an instance we have the field

$$\sigma = (y, -x, z).$$

The Z axis is the singular isolated vector line, the XY plane the singular surface, circles concentric to the origin the singular vector lines in it, and the other vector lines lie on circular cylinders about the Z axis, approaching the XY plane asymptotically.

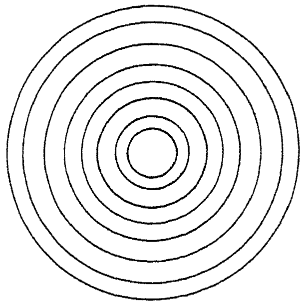

FIG. 9.

The method of determining the character of a singular point will be considered later in connection with the study of the linear vector operator. A singular point at infinity is either a node or a faux.

10. **Singular Lines.** Singularities may not occur alone but may be distributed on lines every point of which is a singular point. This will evidently occur when $\sigma = 0$ gives three surfaces which intersect in a single line. The different types may be arrived at by considering the line of singularities to be straight, and the surfaces of the vector lines with the points of the singular line as singularities to be planes, for the whole problem of the character of the singularities is a problem of analysis situs, and the deformation will not change the character. The types are then as follows:

(1) *Line of Nodes.* Every point of the singular line is a node. A simple example is $\sigma = (x, y, 0)$. The vector lines are all rays passing through the Z axis and parallel to the XY plane.

(2) *Line of Fauces.* There are two singular vector lines through each point of the singular line. As an instance $\sigma = (x, -y, 0)$. The lines through the Z axis parallel to the X and the Y axes are singular, all other vector lines lying on hyperbolic cylinders.

(3) *Line of Foci.* The points of the singular line are approached asymptotically by spirals. As an instance $\sigma = (x + y, y - x, 0)$. The vector lines are logarimithic spirals in planes parallel to the XY plane, wound around the Z axis which is the singular line.

(4) *Line of Centers.* A simple case is $\sigma = (y, -x, 0)$. The vector lines are the Z axis and all circles with it as axis.

11. **Singularities at Infinity.** The character of these is determined by transforming the components of σ so as to bring the regions at infinity into the finite parts of the space we are considering. The asymptotic lines will then have in the transformed space nodes at which the lines are tangent to the asymptotic line.

12. **General Characters.** The problem of the character of a vector field so far as it depends upon the vector lines and their singularities is of great importance. Its general resolution is due to Poincaré. In a series of memoirs in the Journal des Mathématiques* he investigated the qualitative character of the curves which represent the characteristics of differential equations, particularly with the intention of bringing the entire set of integral curves into view at once. Other studies of differential equations usually relate to the character of the functions defined at single points and in their vicinities. The chief difficulty of the more general study is to ascertain the limit cycles. These with the asymptotic division lines separate the field into independent regions.

* Ser. (3) 7 (1881), p. 375; ser. (3) 8 (1882), p. 251; ser. (4) 1 (1885), p. 167. Also Takeo Wado, Mem. Coll. Sci. Tokyo, 2 (1917) 151.

The asymptotic division lines appear on meteorological maps as lines on the surface of the earth towards which, or away from which, the air is moving. They are called in the two cases lines of *convergence*, or lines of *divergence*, respectively. If a division line of this type starts at a node the node may be a point of convergence or a point of divergence. The line will then have the same character. The node in other fields, such as electric or magnetic or heat flow, is a source or a sink. If a division line starts from a faux, the latter is often called a *neutral point*. A focus may be also a point of convergence or point of divergence. In the case of a singular line consisting of foci, the singular line may be a line of convergence or of divergence; in the first case, for instance, the singular line is the core of the *anti-cyclone*, in the latter case, the core of the *cyclone*.

The limit cycles which are not at infinity are division lines which enclose areas that remain isolated in the field. Such phenomena as the *eye of the cyclone* illustrate the occurrence of limit cycles in natural phenomena. The limit cycle may be a line of convergence or a line of divergence, the air in the first case flowing into the line asymptotically from both inside and outside, with the focus serving as a source, and in the other case with conditions reversed.

The practical handling of these problems in meteorological work depends usually upon the isogonal lines: the lines which are loci of equidirected tangents of the vector lines of the field. These are drawn and the infinitesimal tangents drawn across them. The filling in of the vector lines is then a matter of draughtsmanship. The isogonal lines will themselves have singularities and these will enable one to determine somewhat the singularities of the vector lines themselves. Since the unit vector in the direction of σ is constant along an isogon it is evident that

the only change in σ along an isogon is in its intensity, that is, σ keeps the same direction, and its differential is therefore a multiple of σ, that is, the isogons have for their differential equation

$$d\sigma = \sigma dt.$$

Consequently, when $\sigma = 0$ or $\sigma = \infty$ the isogon will have a singular point. It does not follow, however, that all the singular points of the isogons will appear as singular points such as are described above for the vector lines. When the differential equation of the isogons is reduced to the standard form

$$d\rho = \tau du$$

we shall see later that τ will be a linear vector function of σ, and that a linear vector function may have zero directions, so that $\varphi\sigma = 0$, without $\sigma = 0$. Some of the phenomena that may happen are the following, from Bjerknes' *Dynamic Meteorology and Hydrography.* See his plates 42*a*, 42*b*.

1. *Node of Isogons.* These may be positive, in which case the directions of the tangents of the vector lines will increase (that is, the tangent will turn positively) as successive isogons are taken in a positive rotation about the node, or may be negative in the reverse case. The positive node of the isogon will then correspond to a node, a focus, or a center of the vector lines. The negative node of the isogon will correspond to a faux of the vector lines.

If the isogons are parallel, having, therefore, a node at infinity in either of their directions, the vector lines may have asymptotic division lines running in the same direction, or they may have lines of inflexion parallel to the isogons.

2. *Center of Isogons.* When the isogons are cycles they may correspond to very complicated forms of the vector

lines. Several of these are to be found in a paper by Sandstrom, Annalen der Hydrographie und maritimen Meteorologie, vol. 37 (1909), p. 242, Über die Bewegung der Flussigkeiten.

EXERCISES*

* To be solved graphically as far as possible.

1. A translation field is given by $\sigma = (at, bt, ct)$, what are the vector lines, the isogons, and the singularities?

2. A rotation field is given by $\sigma = (mz - ny,\ nx - lz,\ ly - mx)$, what are the isogons, singularities, and vector lines?

3. A field of deformation proportional to the distance in one direction is given by $\sigma = (ax, 0, 0)$. Determine the field.

4. A general field of linear deformation is given by

$$\sigma = (ax + by + cz, fx + gy + hz, kx + ly + mz).$$

determine the various kinds of fields this may represent according to the different possible cases.

5. Consider the quadratic field*

$$\sigma = (x^2 - y^2 - z^2, 2xy, 2xz).$$

6. Consider the quadratic field $\sigma = (xy - xz, yz - yx, zx - zy)$.

7. What are the lines of flow when the motion is stationary in a rotating fluid contained in a cylindrical vessel with vertical axis of rotation?

8. Consider the various fields $\sigma = (ay + x, y - ax, b)$ for different values of a, which is the tangent of the angle between the curves and their polar radii. What happens in the successive diagrams to the isogons, to the curves?

9. Consider the various fields† $\sigma = (1, f(r - a), b)$ where r is the polar radius in the XY plane, a is constant, and f takes the various forms

$$f(x) = x, x^2, x^3, x^{1/2}, x^{1/3}, x^{-1}, x^{-2}, e^x, \log x, \sin x, \tan x.$$

10. Consider the forms $\sigma = (1, f(a\pi \sin r), b)$ where

$$f(x) = \sin x, \cos x, \tan x.$$

11. In various electrical texts, such as Maxwell, *Electricity and Magnetism*, and others, there will be found plates showing the lines of various fields. Discuss these. Also, the meteorological maps in Bjerknes' Dynamic Meteorology, referred to above.

* See Hitchcock, Proc. Amer. Acad. Arts and Sci., 52 (1917), No. 7, pp. 372–454.

† See Sandstrom cited above.

12. In a funnel-shaped vortex of a water-spout the spout may be considered to be made up of twisted funnels, one inside another, the space between the surfaces being a vortex tube. In the Cottage City water-spout, Aug. 19, 1896, the equation of the outside funnel may be taken to be

$$(x^2 + y^2)z = 3600.$$

In this x, y are measured horizontally in meters from the axis of the tubes, and z is measured vertically downwards from the cloud base, which is 1100 meters above the ground. The inner surfaces have the same equation save that instead of 3600 on the right we have $3600/(1.6010)^{2n}$; that is, at any level, the radius of a surface bounding a tube is found from the preceding radius at the same level by dividing by the number whose logarithm (base 10) is 0.20546. From meteorological theory the velocity of the wind on any surface is given by

$$\sigma = (Cr,\ Crz,\ -2Cz)$$

where the first component is the horizontal radial component, the second is the tangential, and the third is the vertical component. C varies for the different surfaces, and is found by multiplying the value for the outside surface by the square of the number 1.6010. In Bigelow's *Atmospheric Radiation, etc.*, p. 200 et seq., is to be found a set of tables for the various values from these data for different levels. Characterize the vortex field of the water-spout.

13. For a dumb-bell-shaped water-spout, likewise, the funnels have the equation

$$(x^2 + y^2) \sin az = \text{const}/A$$

where A varies from surface to surface just as C in the preceding problem. The velocity is given by

$$\sigma = (-Aar \cos az,\ Aar \sin az,\ 2A \sin az),$$

the directions being horizontal radial, tangential and vertical. For the St. Louis tornado, May 27, 1896, the following data are given. Cloud base 1200 meters above the ground, divided into 121 parts called degrees, the ground thus being at 60°, and az being in degrees. The values of A are for the successive funnels

$$0.1573,\ 0.4052,\ 1.0437,\ 2.6883,\ 6.9247,\ 17.837.$$

Characterize the vector lines of this vortex field.

14. In the treatise on The Sun's Radiation, Bigelow gives the following data for a funnel-shaped vortex

$$r^2 z = 6400000/C$$

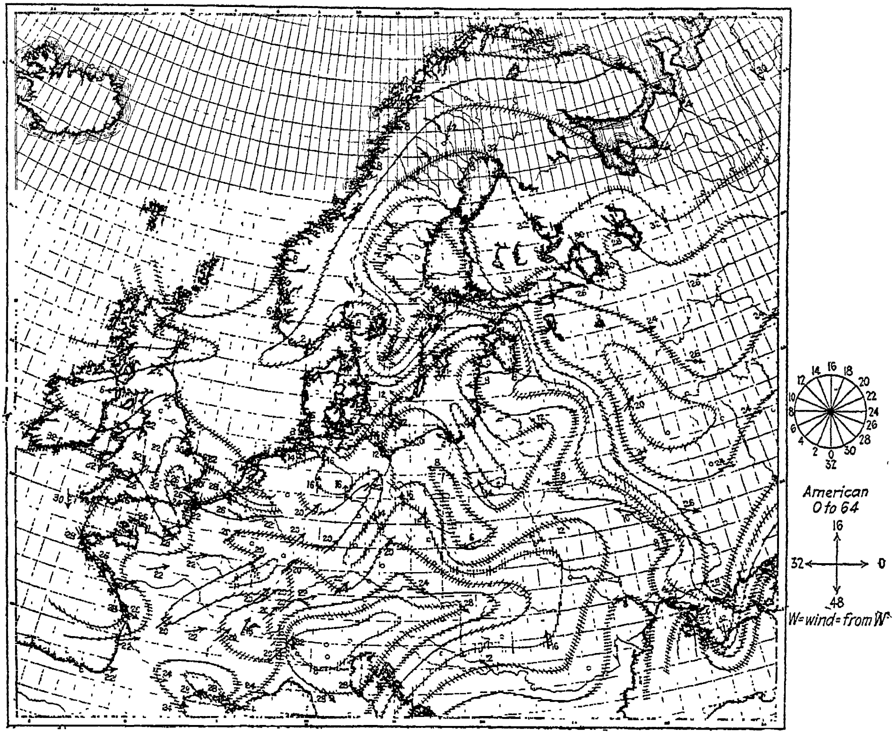

PLATE I

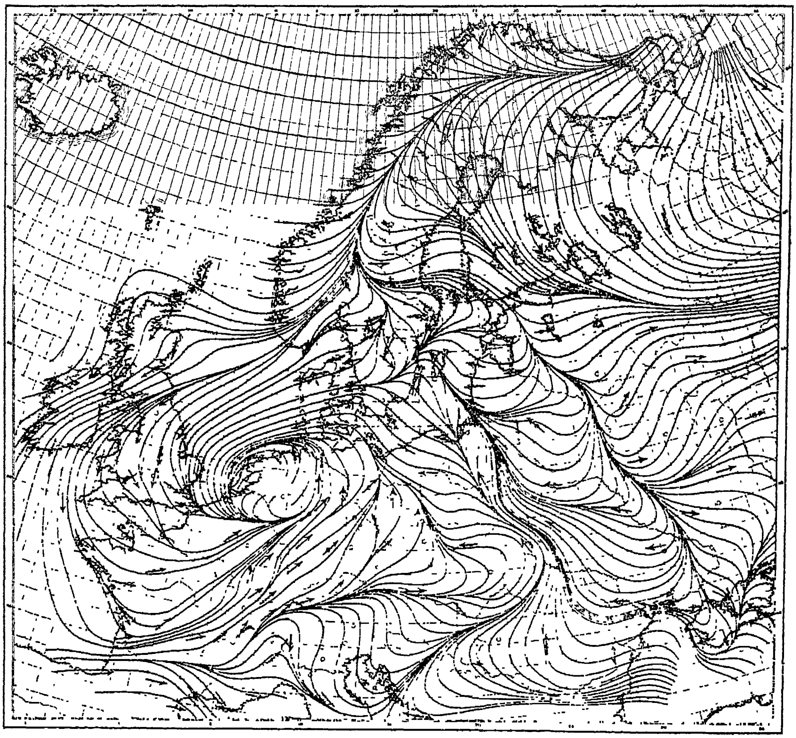

PLATE II

at 500 kilometers $z = 500$, $r = 60474, 26287, 11513, 5023, 2192, 956$.

$$\sigma(Km/\text{sec}) = (Cr,\ Crz,\ -2Cz).$$

Calculate for

$$z = 0,\ 500,\ 1000,\ 2000,\ 5000,\ 10000,\ 20000,\ 30000,\ 40000,\ 50000.$$

The results of the calculations give a vortex field agreeing with Hale's observations.

The vector lines in the last three problems lie on the funnel surfaces, being traced out in fact by a radius rotating about the axis of the vortex, and advancing along the axis according to the law

$$2\theta = -z + C \qquad \text{for the funnel,}$$
$$2\theta = az + C \qquad \text{for the dumb-bell.}$$

15. Study the lines on the plates, which represent on the first plate the isogons for wind velocities, on the second plate the corresponding characteristic lines of wind flow. The date was evening of Jan. 9, 1908. European and American systems of numbering directions are shown in the margin of plate 1. See Sandstrom's paper cited above.

13. **Congruences.** We still have to consider the relations of the various vector lines to each other, noticing that the vector lines constitute geometrically a *congruence*, that is, a two-parameter system of curves in space. The consideration of these matters, however, will have to be postponed to a later chapter.

CHAPTER IV

ADDITION OF VECTORS

1. **Sum of Vectors.** Geometrically, the sum of two or more vectors is found by choosing any one of them as the first, from the terminal point of the first constructing the second (any other), from the terminal point of this constructing the third (any of those left) and so proceeding till all have been successively joined to form a polygon in space with the exception of a final side. If now this last side is constructed by drawing a vector from the initial point of the first to the terminal point of the last, the vector so drawn is called the sum of the several vectors. In case the polygon is already closed the sum is a zero vector. When the sum of two vectors is zero they are said to be *opposite,* and *subtraction* of a vector consists in adding its opposite.

It is evident from the definition that we presuppose a space in which the operations can be effectively carried out. For instance, if the space were curved like a sphere, and the sum of two vectors is found, it would evidently be different according to which is chosen as the first. The study of vector addition in such higher spaces has, however, been considered. *Encyclopédie des sciences mathématiques,* Tome IV, Vol. 2.

2. **Algebraic Sum.** In order to define the sum without reference to space, it is necessary to consider the hypernumbers that are the algebraic representatives of the geometric vectors. We must indeed start with a given set of hypernumbers,

$$\epsilon_1, \epsilon_2, \cdots \epsilon_n,$$

which are the basis of the system of hypernumbers we intend to study. These are sometimes called *imaginaries,* because they are analogous to $\sqrt{-1}$. In the case of three-

dimensional space there are three such hypernumbers in the basis. We combine in thought a numerical value with each of these, the field or domain from which these numerical values are chosen being of great importance. For instance, we may limit our numbers to the domain of integers, the domain of rationals, the domain of reals, or to other more complicated domains, such as certain algebraic fields. We then consider all the multiplexes we can form by putting together into a single entity several of the hypernumbers just formed, as for instance, we would have in three-dimensional space such a compound as

$$\rho = (x\epsilon_1, y\epsilon_2, z\epsilon_3).$$

Since we are now using the base hypernumbers ϵ it is no longer necessary to use the parentheses nor to pay attention to the order of the terms. We drop the use of the comma, however, and substitute the $+$ sign, so that we would now write

$$\rho = x\epsilon_1 + y\epsilon_2 + z\epsilon_3.$$

We may now easily define the algebraic sum of several hypernumbers corresponding to vectors by the formula

$$\rho_i = x_i\epsilon_1 + y_i\epsilon_2 + z_i\epsilon_3, \quad i = 1, 2, \cdots m,$$
$$\sum_{i=1} \rho_i = \Sigma x_i\epsilon_1 + \Sigma y_i\epsilon_2 + \Sigma z_i\epsilon_3.$$

This definition of course includes subtraction as a special case.

It is clear from this definition that to correspond to the geometric definition, it is necessary that the units ϵ correspond to three chosen unit vectors of the space under consideration. They need not be orthogonal, however. The coefficients of the ϵ are then the oblique or rectangular coordinates of the point which terminates the vector if it starts at the origin.

3. **Change of Basis.** We may define all the hypernumbers of the system in terms of a new set linearly related to the original set. For instance, if we write

$$\epsilon_1 = a_{11}\alpha_1 + a_{12}\alpha_2 + a_{13}\alpha_3,$$
$$\epsilon_2 = a_{21}\alpha_1 + a_{22}\alpha_2 + a_{23}\alpha_3,$$
$$\epsilon_3 = a_{31}\alpha_1 + a_{32}\alpha_2 + a_{33}\alpha_3,$$

then ρ becomes

$$\rho = (a_{11}x + a_{21}y + a_{31}z)\alpha_1 \\ + (a_{12}x + a_{22}y + a_{32}z)\alpha_2 + (a_{13}x + a_{23}y + a_{33}z)\alpha_3.$$

It is evident then that if we transform the ϵ's by a non-singular linear homogeneous transformation, the coefficients of the new basis hypernumbers, α, are the transforms of the original coefficients under the *contragredient* transformation.

Inasmuch as the transformation is linear, the transform of a sum will be the sum of the transforms of the terms of the original sum. The transformation as a geometrical process is equivalent to changing the axes. This process evidently gives us a new triple, but must be considered not to give us a new hypernumber nor a new vector. Indeed, a vector cannot be defined by a triple of numbers alone. There is also either explicitly stated or else implicitly understood to be a basis, or on the geometric side a definite set of axes such that the triple gives the components of the vector along these axes. It is evident that the success of any system of vector calculus must then depend upon the choice of modes of combination which are not affected by the change from one basis to another. This is the case with addition as we have defined it. We assume that we may express any vector or hypernumber in terms of any basis we like, and usually the basis will not appear.

If the transformation is such as to leave the angles be-

tween ϵ_1, ϵ_2, ϵ_3 the same as those between α_1, α_2, α_3, the second trihedral being substantially the same as the first rotated into a new position, with the lengths in each case remaining units, then the transformation is called *orthogonal.* We may define an orthogonal transformation algebraically as one such that if followed by the contragredient transformation the original basis is restored.

4. **Differential of a Vector.** If we consider two points at a small distance apart, the vector to one being ρ, to the other ρ', and the vector from the first to the second, $\Delta\rho = \rho' - \rho$, where $\Delta\rho = \Delta s \cdot \epsilon$, ϵ being a unit vector in the direction of the difference, we may then let one point approach the other so that in the limit ϵ takes a definite position, say α, and we may write ds for Δs, and call the result the *differential of* ρ for the given range over which the ρ' runs. In the hypernumbers we likewise arrive at a hypernumber

$$d\rho = dx\epsilon_1 + dy\epsilon_2 + dz\epsilon_3,$$

where now ds is a linear homogeneous irrational function of dx, dy, dz, which $= \sqrt{(dx^2 + dy^2 + dz^2)}$ in case ϵ_1, ϵ_2, ϵ_3 form a trirectangular system of units.

The quotient $d\rho/dt$ is the velocity at the point if t represents the time. The unit vector α is the unit tangent for a curve. We generally represent the principal normal and the binormal by β, γ respectively. When ρ is given as dependent on a single variable parameter, as t for instance, then the ends of ρ may describe a curve. We may have in the algebraic form the coordinates of ρ alone dependent upon the parameter, or we may have both the coordinates and the basis dependent upon t. For instance, we may express ρ in terms of ϵ_1, ϵ_2, ϵ_3 which are not dependent upon t but represent fixed directions geometrically, or we may express ρ in terms of three hypernumbers as ω, τ, ζ which

themselves vary with t, such as the moving axes of a system in space. In relativity theories the latter method of representation plays an important part.

5. **Integral of a Vector.** If we add together n vectors and divide the result by n we have the *mean* of the n vectors, which may be denoted by $\bar{\rho}$. If we select an infinite number of vectors and find the limit of their sum after multiplication by dt, the differential of the parameter by which they are expressed, such limit is called the *integral* of the vector expressed in terms of t, and if we give t two definite values in the integral and subtract one result from the other, the difference is the *integral* of the vector *from the first value of t to the second.* More generally, if we multiply a series of vectors, infinite in number, by a corresponding series of differentials, and find the limit of the sum of the results, such limit, when it exists, is called the integral of the series. In integration, as in differentiation, the usual difficulties met in analysis may appear, but as they are properly difficulties due to the numerical system and not to the hypernumbers, we will suppose that the reader is familiar with the methods of handling them.

The *mean* in the case of a vector which has an infinite sequence of values is the quotient of the integral taken on some set of differentials, divided by the integral of the set of differentials itself. The examples will illustrate the use of the mean.

EXAMPLES

(1) The *centroid* of an arc, an area, or a volume is found by integrating the vector ρ itself multiplied by the differential of the arc, ds, or of the surface, du, or of the volume dv. The integral is then divided by the length of the arc, the area of the surface, or the volume. That is

$$\bar{\rho} = \frac{\int_a^b \rho(s)ds}{b-a} \quad \text{or} \quad \frac{\int\int \rho du}{A} \quad \text{or} \quad \frac{\int\int\int \rho dv}{V}.$$

(2) An example of *average velocity* is found in the following (Bjerknes, *Dynamic Meteorology*, Part II, page 14) observations of a small balloon.

z = Ht. in Meters	Δz	Direction	Velocity (m/sec.)	Products
77				
680	603	S. 50° E.	3.4	2050
960	280	S. 57° E.	4.0	1120
1240	280	S. 36° E.	5.3	1484
1530	290	S. 28° W.	1.5	435
1810	280	S. 2° W.	1.8	504
2090	280	S. 2° W.	2.0	560
2430	340	S. 35° W.	1.5	510
2730	300	S. 53° W.	1.8	540
3040	310	S. 69° W.	1.8	558
3400	360	S. 55° W.	3.0	1080
3710	310	S. 53° W.	2.8	868
4030	320	S. 58° W.	4.4	1408
4400	370	S. 37° W.	10.2	3773

To average the velocities we notice that on the assumption that the upward velocity was uniform the distances vertically can be used to measure the time. We therefore multiply each velocity by the difference of elevations corresponding, the products being set in the last column. These numbers are then taken as the lengths of the vectors whose directions are given by the third column. The sum of these is found graphically, and divided by the total difference of distance upward, that is, 4323. In the same manner we can find graphically the averages for each 1000 meters of ascent. We may now make a new table in order to find other important data, as follows:

Height	Pressure (m-bars)	Dens. (ton/m^3)	Veloc.	Spec. Momentum (ton/m^2 sec.)
4000	622			
3000	705	0.00083	3.8	0.0032
2000	797	0.00092	1.6	0.0015
1000	899	0.00102	2.4	0.0025
75	1003	0.00112	3.7	0.0041

We now find the average velocity between the 1000 m-bar, the 900 m-bar, the 800 m-bar, the 700 m-bar, and the 600 m-bar. The direction is commonly indicated by the integers from 0 to 63 inclusive, the entire circle being divided into 64 parts, each of $5\frac{5}{8}°$. East is 0, North is 16, NW. is 24, etc. The following table is found.

Pressure	Height	Spec. Vol. (m^3/Ton)	Direction	Veloc.	Spec. Momentum
600	4274				
		1217	8	5.2	0.0043
700	3057				
		1087	7	1.7	0.0016
800	1970				
		981	20	2.4	0.0024
900	989				
		890	25	3.7	0.0042
1000	99				
		890	25	3.4	0.0040
1002.6	76				

Of course, specific momenta should be averaged like velocities but usually owing to the rough measurements it is sufficient to find specific momenta from the average velocities.

EXERCISES

1. Average as above the following observations taken at places mentioned (Bjerknes, p. 20), July 25, 1907, at 7 a.m. Greenwich time.

Isobar	Dyn. Ht.	Δz	Direction	Veloc.	
100	16374				*Uccle,* *Lat.* 50° 48′ *Long.* 4° 22′
		4427	0	4.7	
200	11947				
		2627	10	3.2	
300	9320				
		2019	18	3.4	
400	7301				
		1653	19	3.3	
500	5648				
		1408	8	2.6	
600	4240				
		1220	5	2.5	
700	3020				
		1082	4	2.5	
800	1938				
		963	4	1.4	
900	975				
		867	36	4.5	
1000	107				
		9	35	4.5	
1001.2	98				
100	16238				*Zurich,* *Lat.* 47° 23′ *Long.* 8° 33′
		4421	3	10.0	
200	11817				
		2577	6	6.5	
300	9240				
		1992	7	7.6	
400	7248				
		1622	2	10.2	
500	5626				
		1382	3	6.7	
600	4244				
		1206	2	6.8	
700	3038				
		1083	62	5.3	
800	1955				
		978	4	0.6	
900	977				
		506	30	2.1	
955.9	471				
200	11890				*Hamburg,* *Lat.* 53° 33′ *Long.* 9° 59′
		2649	59	9.2	
300	9241				
		2001	57	10.5	
400	7240				
		1597	58	8.8	
500	5643				
		1447	55	8.0	
600	4196				
		1205	49	2.9	
700	2991				
		1064	41	2.9	
800	1927				
		946	38	1.9	
900	981				
		863	56	4.3	
1000	118				
		101	55	3.4	
	17				

2. If the direction of the wind is registered every hour how is the average direction found? Find the average for the following observations.

Station	Pikes Peak		Vienna		Mauritius		Cordoba		S Orkneys	
Elev.	4308 m.		26 m.		15 m.		437 m.		25 m.	
Time	*Summer*		*Winter*		*Dec.–Feb.*		*Winter*		*Summer*	
	Vel.	Az.	Vel.	Az.	Vel.	Az.	Vel.	Az.	Vel.	Az.
0 a.m.	0.84	100°	0.47	62	1.00	100	0.94	115	0.52	70
2	1.34	83	0.56	61	1.30	97	1.06	111	0.52	51
4	1.46	71	0.42	59	1.30	98	1.44	121	0.51	30
6	1.05	57	0.33	57	1.00	119	2.03	132	0.51	5
8	0.43	12	0.22	46	1.10	241	2.17	136	0.52	343
10	0.66	279	0.17	303	1.80	312	0.50	252	0.53	262
12 noon	1.03	262	0.36	257	2.40	326	2.78	314	0.54	255
14	1.09	256	0.58	242	1.60	332	3.56	315	0.54	245
16	0.95	253	0.64	232	1.30	304	3.36	305	0.54	42
18	0.74	247	0.47	223	0.20	10	1.75	299	0.53	35
20	0.49	47	0.14	186	0.90	101	0.72	44	0.53	50
22	0.36	153	0.25	72	1.00	102	0.89	128	0.52	60

Bigelow, *Atmospheric Circulation, etc.*, pp. 313–315.

3. The following table gives the mean magnetic deflecting vectors, in four zones, the intensity measured in 10^{-5} dynes, φ measured from S. to E., N., W., and θ is measured above the horizon. The vector is the deflection from the mean position. Find the average for each zone. (Bigelow, pp. 324–325.)

Time	Arctic			N Temperate			Tropic			S Temperate		
	s	θ	φ	s	θ	φ	s	θ	φ	s	θ	φ
a.m. 0	60	−36°	345°	15	−30°	111°	20	−33°	5°	19	27°	259°
1	63	−44	355	14	−35	109	19	−32	16	19	31	250
2	69	−43	5	14	−32	102	20	−36	7	17	35	251
3	74	−44	16	14	−33	108	20	−42	6	18	36	243
4	75	−42	25	15	−35	112	18	−34	10	20	36	226
5	77	−42	30	17	−33	110	17	−37	6	21	33	223
6	78	−40	32	20	−31	112	19	−36	4	24	31	222
7	76	−40	36	22	− 6	107	21	−37	339	26	24	235
8	65	−37	45	25	3	99	24	−30	297	28	28	248
9	54	−18	68	26	24	66	26	23	228	28	33	256
10	39	31	117	27	37	49	35	25	210	26	−27	296
11	47	44	195	25	38	312	43	22	204	25	−37	327

4. Find the resultant attraction at a point due to a segment of a straight line which is (*a*) of uniform density, (*b*) of density which varies as the square of the distance from one end. What is the mean attraction in each case?

5. Show that $\rho = t\alpha + \frac{1}{2}t^2\beta$ is the equation of a parabola, that the equation of the tangent is $\rho = t_1\alpha + \frac{1}{2}t_1^2\beta + x(\alpha + t_1\beta)$, that tangents from a given point are given by $t = p \pm \sqrt{(p^2 - 2q)}$, the point being $p\alpha + q\beta$, the chord of contact is $\rho = -q\beta + y(\alpha + p\beta)$ which has a direction independent of q so that all points of the line $\rho = p\alpha + z\beta$ have corresponding chords of contact which are parallel. If a chord is to pass through the point $a\alpha + b\beta$ for differing values of ρ, then $q = ap - b$ and the moving point $p\alpha + q\beta$ lies on the line $\rho = p\alpha + (ap - b)\beta$, whose direction is independent of b.

6. If α, β, γ are vectors to three collinear points, then we can find three numbers a, b, c such that

$$a\alpha + b\beta + c\gamma = 0 = a + b + c.$$

7. In problem 5 show that if three points are taken on the parabola corresponding to the values t_1, t_2, t_3, then the three points of intersection of the sides of the triangle they determine with the tangents at the vertices of the triangle are collinear.

8. Determine the points that divide the segment joining A and B, points with vectors α and β, in the ratio $l : m$, both internally and externally. Apply the result to find the polar of a point with respect to a given triangle, that is, the line which passes through the three points that are harmonic on the three sides respectively with the intersection of a line through the given point and the vertex opposite the side.

9. Show how to find the resultant field due to superimposed fields.

10. A curve on a surface is given by $\rho = u(u, v)$, $u = f(v)$, study the differential of ρ.

CHAPTER V

VECTORS IN A PLANE

1. **Ratio of Two Vectors.** We purpose in this chapter to make a more detailed study of vectors in a plane and the hypernumbers corresponding. In the plane it is convenient to take some assigned unit vector as a reference for all others in the plane, though this is not at all necessary in most problems. In fact we go back for a moment to the fundamental idea underlying the metric notion of number. According to this a number is defined to be the ratio between two quantities of the same concrete kind, such as the ratio of a rod to a foot. If now we consider the ratio of vectors, regarding them as the same kind of quantity, it is clear that the ratio will involve more than merely numerical ratio of lengths. The ratio in this case is in fact what we have called a hypernumber. For every pair of vectors ρ, π there exists a ratio $\rho : \pi$ and a reciprocal ratio $\pi : \rho$. This ratio we will designate by a roman character

$$q = \rho : \pi = \rho/\pi = \frac{\rho}{\pi} \cdot$$

That is to say, we may substitute ρ for $q\pi$.

2. **Complex Numbers.** If we draw ρ and π from one point, they will form a figure which has two segments for sides and an angle. (In case they coincide we still consider they have an angle, namely zero.) In this figure ρ is the initial side and π is the terminal side. Then their complex ratio is $\pi : \rho$. Since this ratio is to be looked upon as a multiplier, it is clear that if we were to reduce the sides in the same proportion, the ratio would not be changed.

A change of angle would, however, give a different ratio. However, we will agree that all ratios are to be considered as equivalent, or as we shall usually say, equal, not only when the figures to which they correspond have sides in the same proportion, but also when they have the same angles and sides in proportion, even if not placed in the plane in the same position. For instance, if the vectors AB, AC make a triangle which is similar to the triangle DE, DF, if we take the sides in this order, then we shall consider that whatever complex or hypernumber multiplies AC into AB will also multiply DF into DE. This axiom of equivalence is not only important but it differentiates this particular hypernumber from others which might just as well be taken as fundamental. For instance, the Gibbs dyad of $\pi : \rho$ is equally a hypernumber, but we cannot substitute for π or ρ any other vectors than mere multiples of π or ρ. It is clear that in the Gibbs dyad we have a more restricted hypernumber than in the ordinary complex number which has been just defined, and which is a special case of the Hamiltonian quaternion. If we have a Gibbs dyad q, we can find the two vectors π and ρ save as to their actual lengths. But with the complex number q we cannot find π and ρ further than to say that for every vector there is another in the ratio q. In other words the only transformations allowed in the Gibbs dyad are translation of the figure AB, AC or magnification of it. In the Hamiltonian quaternion, or complex number, the transformations of the figure AB, AC may be not only those just mentioned but rotation in the plane.

In order to find a satisfactory form for the hypernumber q which we have characterized, we further notice that if we change the length of π in the ratio m then we must change q in the same ratio, and if we set for the ratio of the

length or intensity of π to that of ρ the number r, it is evident that we ought to take for q an expression of the form

$$q = r\varphi(\theta),$$

where $\varphi(\theta)$ is a function of θ, the angle between ρ and π, only. Further if we notice that we now have

$$\pi = r\varphi(\theta)\rho,$$

where the first factor affects the change of length, the second the change of direction, it is plain that for a second multiplication by another complex number $q' = r'\varphi(\theta')$, we should have

$$\pi' = r'r\varphi(\theta')\varphi(\theta)\rho = r'r\varphi(\theta' + \theta)\rho.$$

Whence we must consider that

$$\varphi(\theta')\varphi(\theta) = \varphi(\theta' + \theta) = \varphi(\theta)\varphi(\theta').$$

These expressions are functions of two ordinary numerical parameters, θ, θ', and are subject to partial differentiation, just like any other expressions. Differentiating first as to θ, then as to θ', we find (φ' being the derivative)

$$\varphi'(\theta)\varphi(\theta') = \varphi'(\theta + \theta') = \varphi'(\theta')\varphi(\theta),$$

whence

$$\frac{\varphi'(\theta)}{\varphi(\theta)} = \frac{\varphi'(\theta')}{\varphi(\theta')} = k,$$

where k is a constant and does not depend upon the angle at all. It may, however, depend upon the plane in which the vectors lie, so that for different planes k may be, and in fact is, different.

Since, when $\theta = 0$ the hypernumber becomes a mere numerical multiplier,

$$\varphi'(0) = k\varphi(0).$$

If now we examine the particular function

$$\varphi(\theta) = \cos\theta + k\sin\theta,$$

which gives

$$\varphi'(\theta) = -\sin\theta + k\cos\theta = k\cos\theta + k^2\sin\theta,$$

we find all conditions are satisfied if we take $k^2 = -1$. We may then properly use this function to define φ. This very simple condition then enables us to define hypernumbers of this kind, so that we write

$$q = r(\cos\theta + k\sin\theta) = r\ \mathrm{cks}\ \theta = r_\theta,$$

where $k^2 = -1$.

3. **Imaginaries.** It is desirable to notice carefully here that we must take k^2 equal to -1, the same negative number that we have always been using. This is important because there are other points of view from which the character of k and k^2 would be differently regarded. For instance, in the original paper of Hamilton, *On Algebraic Couples*, the k, or its equivalent, is regarded as a linear substitution or operator, which converts the couple (a, b) into the couple $(-b, a)$. While it is true that we may so regard the imaginary, it becomes at once obvious that we must then draw distinctions between 1 as an operator, and 1 as a number, and so for -1, and indeed for any expression $x + yi$. In fact, such distinctions are drawn, and we find these operators occasionally called *matrix unity*, etc. From the point of view of the hypernumber, this distinction is not possible. Hypernumbers are extensions of the number system, similar to radicals and other algebraic numbers. The fact that, as we will see later, they are not in general commutative, does not prevent their being an extension.

4. **Real, Imaginary, Tensor, Versor.** In the complex number

$$q = r\cos\theta + r\sin\theta\cdot k$$

the term $r\cos\theta$ is called the *real* part of q and may be written Rq. The term $r\sin\theta\cdot k$ is called the *imaginary* part of q

and written Iq. The number r is called the *tensor* of q and written Tq. The expression $\cos\theta + \sin\theta\cdot k$ is called the *versor* of q and written Uq. Therefore,

$$q = Rq + Iq = TqUq.$$

If q appears in the form $q = a + bk$ we see at once that

$$Rq = a, \qquad Iq = bk, \qquad Tq = \sqrt{(a^2 + b^2)}, \qquad \theta = \tan^{-1}b/a.$$

5. **Division.** If we have $\pi = q\rho$, then we also write $\rho = q^{-1}\pi$. It becomes evident that

$$Rq^{-1} = Rq/(Tq)^2, \quad Iq^{-1} = -\,Iq/(Tq)^2, \quad Tq^{-1} = 1/Tq,$$
$$Uq^{-1} = \cos\theta - \sin\theta\cdot k.$$

6. **Conjugate, Norm.** The hypernumber $\bar{q} = Kq = Rq - Iq$ is called the *conjugate of* q. If q belongs to the figure AB, AC, then $\bar{q}$ belongs to an inversely similar triangle, that is, a similar triangle which has been reflected in some straight line of the plane. The product $q\bar{q} = Nq = (Tq)^2$ is called the *norm* of q. It also has the name *modulus* of q, particularly in the theory of functions of complex variables.

Evidently,

$$Rq = \tfrac{1}{2}(q + \bar{q}), \quad Iq = \tfrac{1}{2}(q - \bar{q}), \quad q^{-1} = \bar{q}/Nq, \quad Uq^{-1} = U\bar{q}.$$

7. **Products of Complex Numbers.** From the definitions it is clear that the product of two complex numbers q, r, is a complex number s, such that

$$\begin{aligned}
Ts &= TqTr, \qquad \angle s = \angle q + \angle r,\\
Rqr &= Rrq = R\overline{qr} = R\overline{rq} = RqRr - TIqIr,\\
Rq\bar{r} &= R\bar{q}r = R\bar{r}q = Rr\bar{q} = RqRr + TIqIr,\\
Iqr &= Irq = -\,I\overline{qr} = -\,I\overline{rq} = RqIr + RrIq,\\
Iq\bar{r} &= I\bar{r}q = -\,I\bar{q}r = -\,Ir\bar{q} = RrIq - RqIr.
\end{aligned}$$

Hence if $Rqr = 0$, the angles of q and r are complementary or have 270° for their sum.

If $\quad R q\bar{r} = 0$, the angles differ by 90°. In particular we may take $r = 1$.

If $\quad Iqr = 0$, the angles are supplementary or opposite.

If $\quad Iq\bar{r} = 0$, the angles are equal or differ by 180°.

8. **Continued Products.** We need only to notice that

$$(\overline{qrs\cdots z}) = (\bar{z}\cdots\bar{s}\bar{r}\bar{q}).$$

It is not really necessary to reverse the order here as the products are commutative, but in quaternions, of which these numbers are particular cases, the products are not usually commutative, and the order must be as here written.

9. **Triangles.** If β, γ, δ, ϵ are vectors in the plane, and

$$\epsilon = q\beta, \qquad \delta = q\gamma,$$

then the triangle of β, ϵ is similar to that of γ, δ, while if

$$\epsilon = q\beta, \qquad \delta = \bar{q}\gamma,$$

the triangles are inversely similar.

These equations enable us to apply complex numbers to certain classes of problems with great success.

10. **Use of Complex Numbers as Vectors.** If a vector α is taken as unit, every vector in the plane may be written in the form $q\alpha$, for some properly chosen q. We may therefore dispense with the writing of the α, and talk of the vector q, always with the implied reference to a certain unit α. This is the well-known method of Wessel, Argand, Gauss, and others. However, it should be noticed that we have no occasion to talk of q as a point in the plane.

EXAMPLES

(1) Calculate the path of the steam in a two-wheel turbine from the following data. The two wheels are rigidly connected and rotate with a speed $\alpha = 400_{0^\circ}$ ft./sec. Be-

tween them are stationary buckets which turn the exhaust steam from the buckets of the first wheel into those of the second wheel. The friction in each bucket reduces the speed by 12%. The steam issues from the expansion nozzle at a speed of $\beta = 2200_{20^\circ}$. The proper exhaust angles of the buckets are 24°, 30°, 45°. Find the proper entrance angles of the buckets.

$\gamma =$ relative velocity of steam at entrance to first wheel.
$= 2200_{20} - 400_0 = 1830_{24.3}$.
$\delta =$ velocity of issuing steam, 88% of preceding,
$= 1610_{156}$.
$\epsilon =$ entrance velocity to stationary bucket.
$= \delta + \alpha = 1610_{156} + 400_0 = 1255_{148.4}$.
$\zeta =$ exit $= 1105_{30}$.
$\theta =$ entrance to next bucket $= \zeta - \alpha = 1105_{30} - 400_0$
$= 780_{44.3}$.
$\eta =$ exit $= 690_{135}$. Absolute exit velocity $= 690_{135}$
$+ 400_0 = 495_{100}$.

Steinmetz, *Engineering Mathematics.*

(2). We may suppose the student is somewhat familiar with the usual elementary theory of the functions of a complex variable. If w is an analytic function of z, both complex numbers, then the real part of w, Rw, considered as a function of x, y or u, v, the two parameters which determine z, will give a system of curves in the x, y, or the u, v plane. These may be considered to be the transformations of the curves $Rw =$ const. which are straight lines parallel to the Y axis in the w plane. Similarly for the imaginary part. The two sets will be orthogonal to each other, since the slope of the first set will be $-\dfrac{\partial Rw}{\partial x}\bigg/\dfrac{\partial Rw}{\partial y}$; and of the other set $-\dfrac{\partial TIw}{\partial x}\bigg/\dfrac{\partial TIw}{\partial y}$. But these are

negative reciprocal, since

$$\frac{\partial Rw}{\partial x} = \frac{\partial TIw}{\partial y} \qquad \text{and} \qquad \frac{\partial Rw}{\partial y} = -\frac{\partial TIw}{\partial x}.$$

EXERCISES

1. If a particle is moving with the velocity $120_{25°}$ and enters a medium which has a velocity given by

$$\sigma = \rho + 36 \sin \angle [\rho, 0]_{8°},$$

what will be its path?

2. The wind is blowing steadily from the northwest at a rate of 16 ft./sec. A boat is carried round in circles with a velocity 12 ft./sec. divided by the distance from the center. The two velocities are compounded, find the motion of the boat if it starts at the point $\rho = 4_{0°}$.

3. A slow stream flows in at the point $12_{0°}$ and out at the point $12_{180°}$, the lines of flow being circles and the speed constant. A chip is floating on the stream and is blown by the wind with a velocity $6_{40°}$. Find its path.

4. If a triangle is made with the sides q, r then $R.q\bar{r}$ is the *power* of the vertex with reference to the circle whose diameter is the opposite side. The area of the triangle is $\frac{1}{2}TIq\bar{r}$.

5. The sum $q + r$ can be found by drawing vectors $q\alpha$, $r\alpha$.

6. How is qr constructed? $q\bar{r}$?

7. If OAE is a straight line and OCF another, and if EC and AF intersect in B, then $OA \cdot BC + OC \cdot AB + OB \cdot CA = 0$. If O, A, B, C are concyclic this gives Ptolemy's theorem.

8. If ABC is a triangle and LM a segment, and if we construct LMP similar to ABC, LMQ similar to BCA, and LMR similar to CAB, then PQR is similar to CAB.

9. If the variable complex number u depends on the real number x as a variable parameter, by the linear fractional form

$$u = \frac{ax + b}{cx + d},$$

then for different values of x the vector representing u will terminate on a circle.

For if we construct

$$w = \frac{u - \frac{b}{d}}{u - \frac{a}{c}}$$

this reduces to $-(cx/d)$, hence the angle of w, which is the angle between $u - a/c$ and $u - b/d$, is the angle of $-d/c$ and is therefore constant. Hence the circle goes through a/c ($x = \infty$) and b/d ($x = 0$).

10. If

$$u = \frac{x(c - b)a + b(a - c)}{k(c - b) + (a - c)},$$

where x is a variable real parameter, then the vector representative of u will terminate on the circle through A, B, C, where OA represents a, OB represents b, and OC represents c.

11. Given three circles with centers C_1, C_2, C_3, and O their radical center, P any point in the plane, then the differences of the powers of P with respect to the three pairs of circles are proportional respectively to the projections of the sides of the triangle $C_1C_2C_3$ on OP.

12. Construct a polygon of n sides when there is given a set of points, $C_1, C_2, \cdots, C_n$ which divide the sides in given ratios $a_1 : b_1, a_2 : b_2, \cdots, a_n : b_n$.

If the vertices are $\xi_1, \xi_2, \cdots, \xi_n$, and the points $C_1, C_2, \cdots, C_n$ are at the ends of vectors $\gamma_1, \gamma_2, \cdots, \gamma_n$, we have

$$a_1\xi_1 + b_1\xi_2 = \gamma_1(a_1 + b_1) \quad \cdots \quad a_n\xi_n + b_n\xi_1 = \gamma_n(a_n + b_n).$$

The solution of these equations will locate the vertices. When is the solution ambiguous or impossible?

13. Construct two directly similar triangles whose bases are given vectors in the plane, fixed in position, so that the two triangles have a common vertex.

14. Construct the common vertex of two inversely similar triangles whose bases are given.

15. Construct a triangle ABC when the lengths of the sides AB and AC are given and the length of the bisector AD.

16. Construct a triangle XYZ directly similar to a given triangle PQR whose vertices shall be at given distances from a fixed point O.

Let the length of OX be a, of OY be b, and of OZ be c. Then X is anywhere on the circle of radius a and center O. We have $XY/XZ = PQ/PR$, that is,

$$\frac{OY - OX}{OZ - OX} = \frac{PQ}{PR},$$

whence we have

$$OX \cdot QR + OY \cdot RP + OZ \cdot PQ = 0.$$

We draw OXK directly similar to RPQ giving $KO/OX = QR/RP$ and $KO + OY + OZ \dfrac{PQ}{RP} = 0$, that is,

$$YK = OZ \frac{PQ}{RP}.$$

In KOY we have the base KO and the length $OY = b$, and length of

$$YK = c \frac{\text{length } PQ}{\text{length } RP}.$$

We can therefore construct KOY and the problem is solved.

17. *The hydrographic problem.* Find a point X from which the three sides of a given triangle ABC are seen under given angles.

$XB/XA = y \text{ cks } \theta$, $XC/XA = z \text{ cks } \varphi$.

$$XB = XA + AB, \ XC = XA + AC.$$

Eliminate XA giving $z \text{ cks } \varphi \cdot BA + y \text{ cks } \theta \cdot AC = BC$. Find U such that $\angle ABU = \angle AXC$, $\angle ACU = \angle AXB$, then $BU = z \text{ cks } \varphi \cdot BA$, $CU = y \text{ cks } \theta\, CA$.

Construct $\Delta\, ACX$ directly similar to $\Delta\, AUB$.

18. Find the condition that the three lines perpendicular to the three vectors $p\alpha$, $q\alpha$, $r\alpha$ at their extremities be concurrent.

We have $p + xkp = q + ykq = r + zkr$. Taking conjugates $\bar{q} - xk\bar{p} = \bar{p} - yk\bar{q} = \bar{r} - zk\bar{r}$. Eliminate x, y, z from the four equations.

19. If a ray at angle β is reflected in a mirror at angle α the reflected ray is in the direction whose angle is $2\alpha - \beta$. Study a chain of mirrors. Show that the final direction is independent of some of the angles.

20. Show that if the normal to a line is α and a point P is distant y from the line, and from P as a source of light a ray is reflected from the line, its initial direction being $-q\alpha$, then the reflected ray has for equation $-2y\alpha + t\bar{q}\alpha = \rho$.

For further study along these lines, see Laisant: *Théorie et Application des Équipollences.*

11. **Alternating Currents.** We will notice an application of these hypernumbers to the theory of alternating currents and electromotive forces, due to C. P. Steinmetz.

If an alternating current is given by the equation

$$I = I_0 \cos 2\pi f(t - t_1),$$

the graph of the current in terms of t is a circle whose diameter is I_0 making an angle with the position for $t = 0$ of $2\pi f t_1$. The angle is called the *phase angle* of the current. If two such currents of the same frequency are superim-

posed on the same circuit, say

$$I = I_0 \cos 2\pi f(t - t_1),$$
$$I' = I_0' \cos 2\pi f(t - t_1'),$$

we may set

$$I_0 \cos 2\pi f t_1 + I_0' \cos 2\pi f t_1' = I_0'' \cos 2\pi f t_2.$$
$$I_0 \sin 2\pi f t_1 + I_0' \sin 2\pi f t_1' = I_0'' \sin 2\pi f t_2,$$
$$I'' = I_0'' \cos 2\pi f(t - t_2),$$

which also has for its graph a circle, whose diameter is the vector sum of the diameters of the other two circles. We may then fairly represent alternating currents of the simple type and of the same frequency by the vectors which are the diameters of the corresponding circles. The same may be said of the electromotive forces.

If we represent the current and the electromotive force on the same diagram, the current would be indicated by a yellow vector (let us say) traveling around the origin, with its extremity on its circle, while at the same time the electromotive force would be represented by a blue vector traveling with the same angular speed around a circle with a diameter of different length perhaps. The yellow and the blue vectors would generally not coincide, but they would maintain an invariable angle, hence, if each is considered to be represented by a vector, the ratio of these vectors would be such that its angle would be the same for all times. This angle is called the angle of lag, or lead, according as the E.M.F. is behind the current or ahead of it.

The law connecting the vectors is

$$E = ZI,$$

where E is the electromotive force vector, that is, the vector diameter of its circle, I is the current vector, the diameter of its circle, and Z is a hypernumber called the *impedance,*

$[\Phi/\Theta]$, measured in *ohms*. The scalar part of Z is the resistance of the circuit, while the imaginary part is the reactance, the formula for Z being

$$Z = r - xk.$$

The value of x is $2\pi fL$, where f is the frequency, $[T^{-1}]$, and L is the inductance, $[\Phi\Theta^{-1}T]$, in *henrys*, or $-1/2\pi fC$ where C is the *permittance*, $[\Theta T\Phi^{-1}]$, in *farads*. [1 *farad* $= 9 \cdot 10^{11}$ *e.s. units* $= 10^{-9}$ *e.m. units, and* 1 *henry* $= \frac{1}{9}10^{-11}$ *e.s. units* $= 10^{9}$ *e.m. units.*] It is to be noticed that reactance due to the capacity of the circuit is opposite in sign to that due to inductance.

The law above is called the generalized Ohm's law. We may also generalize Kirchoff's laws, the two generalizations being due to Steinmetz, and having the highest importance, inasmuch as by the use of these hypernumbers the same type of calculation may be used on alternating circuits as on direct circuits. The generalization of Kirchoff's laws is as follows:

(1) The vector sum of all electromotive forces acting in a closed circuit is zero, if resistance and reactance electromotive forces are counted as counter electromotive forces.

(2) The vector sum of all currents directed toward a distributing point is zero.

(3) In a number of impedances in series the joint impedance is the vector sum of all the impedances, but in a parallel connected circuit the joint admittance (reciprocal of impedance) is the sum of the several admittances.

The impedance gives the angle of lag or lead, as the angle of a hypernumber of this type.

We desire to emphasize the fact that in impedances we have physical cases of complex numbers. They involve complex numbers just as much as velocities involve positive

or negative velocity, or rotations involve positive or negative. We may also affirm that the complex currents and electromotive forces are real physical existences, every current implying a power current and a wattless current whose values lag 90° (as time) behind the power current. The power electromotive force is merely the real part of the complex electromotive force, and the wattless E.M.F. the imaginary part of the complex electromotive force, both being given by the complex current and the complex impedance.

We find at the different points of a transmission line that the complex current and complex electromotive force satisfy the differential equations

$$dI/ds = (g + C\omega k)E, \qquad dE/ds = (r + L\omega k)I.$$

The letters stand for quantities as follows: g is *mhos/mile*, r is *ohms/mile*, C is *farads/mile*, L is *henrys/mile*. $\omega = 2\pi f$.

Setting

$$m^2 = (r + L\omega k)(g + C\pi k), \quad l^2 = (r + L\omega k)/(g + C\omega k),$$

so that m is $[L^{-1}]$ while l is ohms/mile, the solution of the equations is

$$E = E_0 \cosh ms + lI_0 \sinh ms,$$
$$I = I_0 \cosh ms + l^{-1}E_0 \sinh ms,$$

where E_0 and I_0 are the initial values, that is, where $s = 0$. If we set $E_0 = Z_0 I_0$ and then set $Z_0 = Z \cosh h$, $l = Z \sinh h$ we have

$$E = Z \cosh (ms + h)I_0, \qquad I = l^{-1}Z \sinh (ms + h)I_0,$$

$$E = l \coth (ms + h)I,$$
$$E = \operatorname{sech} h \cosh (ms + h)E_0,$$
$$I = \operatorname{csch} h \sinh (ms + h)I_0.$$

To find where the wattless current of the initial station has become the power current we set $I = kI_0$, that is,

$$\sinh (ms + h) = k \sinh h.$$

The value of s must be real.

EXAMPLES

(1) Let $r = 2$ ohms/mile, $L = 0.02$ henrys/mile,
$C = 0.0000005$ farads/mile,

$g = 0$, $\omega = 2000$, $\omega L = 40$ ohms/mile, conductor reactance,

$r + L\omega k = 2 + 40k$ ohms/mile impedance $= 40.5_{87.15°}$.

$\omega C = 0.001$ mhos/mile dielectric susceptance.

$g + C\omega k = 0.001k$ mhos/mile dielectric admittance $= 0.001_{90°}$.

$(g + C\omega k)^{-1} = 1000k^{-1} = 1000_{270°}$ ohms/mile dielectric impedance.

$m^2 = 0.0405_{177.15°}$, $\quad m = 0.2001_{88.58°}$,

$l^2 = 40500_{-2.85°}$, $\quad l = 201.25_{-1.43°}$.

Let the values at the receiver ($s = 0$) be

$$E_0 = 1000_{0°} \text{ volts}, \; I_0 = 0_{0°}.$$

Then we have $\quad E = 1000 \cosh s0.2001_{88.58°}$,

for $s = 100$ $\quad E = 1000 \cosh 20.01_{88.58} = 625.9_{45.92°}$,

$\quad I = 2.77_{27°}$,

for $s = 8$ $\quad E = 50.01_{126.85°}$,

for $s = 16$ $\quad E = 1001_{180.3°}$,

for $s = 15.7$ $\quad E = 1000_{180°}$, a reversal of phase,

for $s = 7.85$ $\quad E = 0_{90°}$.

At points distant 31.4 miles the values are the same.

If we assume that at the receiver end a current is to be maintained with

$$I_0 = 50_{40°} \qquad \text{with} \qquad E_0 = 1000_{0°},$$

$E = 1000 \cosh s0.2001_{88.58°} + 10062_{38.57°} \sinh s0.2001_{88.58°}$,

$I = 50_{40°} \cosh sm + 5_{1.43°} \sinh sm$.

At $s = 100$ $E = 10730_{11355°}$.

MacMahon, *Hyperbolic Functions.*

(2) Let $E_0 = 10000$, $I_0 = 65_{13.5°}$ $r = 1$, $g = 0.00002$
$C\omega = 0.00020$ period 221.5 miles, $\omega L = 4$.

(3) The product $P = \bar{E}I$ represents the power of the alternating current, with the understanding that the frequency is doubled. The real or scalar part is the effective power, the imaginary part the wattless or reactive power. The value of TP is the total apparent power. The cos $\angle P$ is the power factor, and sin $\angle P$ is the induction factor. The torque, which is the product of the magnetic flux by the armature magnetomotive force times the sine of their angle is proportional to $TI \cdot P$, where E is the generated electromotive force, and I is the secondary current. In fact, the torque is $TI \cdot E\bar{I} \cdot p/2\pi f$ where p is the number of poles (pairs) of the motor.

12. Divergence and Curl. In a general vector field the lines have relations to one another, besides having the peculiarities of the singularities of the field. The most important of these relations depend upon the way the lines approach one another, and the shape and position of a moving cross-section of a vector tube. There is also at each point of the field an intensity of the field as well as a direction, and this will change from point to point.

Divergence of Plane Lines. If we examine the drawing of the field of a vector distribution in a plane, we may easily measure the rate of approach of neighboring lines. Starting from two points, one on each line, at the intersection of the normal at a point of the first line and the second line, we follow the two lines measuring the distance apart on a normal from the first. The rate of increase of this normal distance divided by the normal distance and the distance traveled from the initial point is the *divergence of the lines*, or as we shall say briefly the *geometric divergence* of the field. It is easily seen that in this case of a plane

field it is merely the curvature of the curves orthogonal to the curves of the field.

For instance, in the figure, the tangent to a curve of the field is α, the normal at the same point β. The neighboring curve goes through C. The differential of the normal, which is the difference of BD and AC, divided by AC, or BD, is the rate of divergence of the second curve from the first for the distance AB. Hence, if we also divide by AB we will have the rate of angular turn of the tangent α in moving to the neighboring curve, the one from C. This rate of angular turn of the tangent of the field is the same as the rate of turn of the normal of the orthogonal system, and is thus the curvature of the normal system.

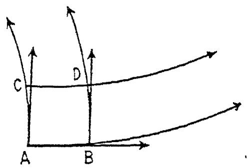

FIG. 10.

Curl of Plane Lines. If we find the curvature of the original lines of the field we have a quantity of much importance, which may be called the *geometric curl.* This must be taken plus when the normal to the field on the convex side of the curve makes a positive right angle with the tangent, and negative when it makes a negative right angle with the tangent. Curl is really a vector, but for the case of a plane field the direction would be perpendicular to the plane for the curl at every point, and we may consider only its intensity.

Divergence of Field. Since the field has an intensity as well as a direction, let the vector characterizing the field be $\sigma = T\sigma \cdot \alpha$. Then the rate of change of $T\sigma$ in the direction of α, the tangent, is represented by $d_a T\sigma$. Let us now consider an elementary area between two neighboring curves of the field, and two neighboring normals. If we consider $T\sigma$ as an intensity of some quantity whose amount

depends also upon the length of the infinitesimal normal curve, so that we consider the quantity $T\sigma \cdot dn$, then the value of this quantity, which we will call the *transport* of the differential tube (strip in the case of a plane field), $T\sigma$ being the density of transport, will vary for different cross-sections of the tube, and for the case under consideration, would be $T\sigma' dn' - T\sigma dn$. But $T\sigma' = T\sigma + d_\alpha T\sigma \cdot ds$ and $dn' = dn + ds \cdot dn$ times the divergence of the lines. Therefore, the differential of the transport will be (to terms of the first order) $ds \times dn \times (T\sigma$ times divergence $+ d_\alpha T\sigma)$. Hence, the density of this rate of change of the transport is $T\sigma$ times the divergence + the rate of change of $T\sigma$ along the tangent of the vector line of the field. This quantity we call the *divergence of the field* at the initial point, and sometimes it will be indicated by div. σ, sometimes by $-S\nabla\sigma$, a notation which will be explained. It is clear that if the lines of a field are perpendicular to a set of straight lines, since the curvature of the straight lines is zero, the divergence of the original lines is zero, and the expression reduces to $d_\alpha T\sigma$.

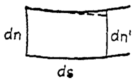

FIG. 11.

Curl of Field. We may also study the *circulation* of the vector σ along its lines, by which we mean the product of the intensity $T\sigma$ by a differential arc, that is, $T\sigma ds$. On the neighboring vector line there is a different intensity, $T\sigma'$, and a different differential arc ds'. The differential of the circulation is easily found in the same manner as the divergence, and turns out to be

$$-(d_\beta T\sigma + T\sigma \times \text{curl of the vector lines}).$$

This quantity we shall call the *curl* of the field, written sometimes curl σ, and more frequently $V\nabla\sigma$, which notation will be explained.

It is evident that the curl of σ is the line integral of the $T\sigma ds$ around the elementary area, for the parts contributed by the boundary normal to the field will be zero. Hence, we may say that curl σ is the limit of the circulation of σ around an elementary area constructed as above, to the area enclosed. We will see later that the shape of the area is not material.

Likewise, the divergence is clearly the ratio to the elementary area of the line integral of the normal component of σ along the path of integration. We will see that this also is independent of the shape of the area.

Further, we see that in a field in which the intensity of σ is constant the divergence becomes the geometric divergence times the intensity $T\sigma$, and the curl becomes the geometric curl times the intensity $T\sigma$.

Divergence and curl have many applications in vector analysis in its applications to geometry and physics. These appear particularly in the applications to space. A simple example of convergence or divergence is shown in the changing density of a gas moving over a plane. A simple case of curl is shown by a needle imbedded in a moving viscous fluid. The angular rate of turn of the direction of the needle is one-half the curl of the velocity.

13. **Lines as Levels.** If the general equation of a given set of curves is

$$u(x, y) = c,$$

these curves will be the vector lines of an infinity of fields, for if the differential equation of the lines is

$$dx/X = dy/Y,$$

then we must have

$$X\partial u/\partial x + Y\partial u/\partial y = 0$$

and for the field

$$\sigma = X\alpha + Y\beta.$$

We may evidently choose X arbitrarily and then find Y uniquely from the equation. However, if σ_1 is any one field so determined, any other field is of the form

$$\sigma = \sigma_1 R(x, y).$$

The orthogonal set of curves would have for their finite equation

$$v(x, y) = c$$

and for their differential equation

$$X\partial v/\partial y - Y\partial v/\partial x = 0.$$

If we use α uniformly to represent the unit tangent of the u set, and β the unit tangent of the v set, then $\beta = k\alpha$. The gradient of the function u is then $d_\beta u \cdot \beta$, and the gradient of the function v is $-d_\alpha v \cdot \alpha$. But the gradient of u is also (u_x, u_y) and of v is $(v_x, v_y) = (u_y, -u_x)$. It follows that the tensors of the gradients are equal. In fact, writing ∇u for gradient u, we have $\nabla v = k\nabla u$. We also have for whatever fields belong to the two sets of orthogonal lines for u curves, $\sigma = r\nabla v$, for the v curves, $\sigma' = s\nabla u$, or also we may write

$$\nabla v = t\alpha, \qquad \nabla u = t\beta, \qquad \sigma = T\sigma \cdot \alpha.$$

14. **Nabla.** The symbol ∇ is called *nabla*, and evidently may be written in the form $\alpha\partial/\partial x + \beta\partial/\partial y$ for vectors in a plane. We will see later that for vectors in space it may be written $\alpha\partial/\partial x + \beta\partial/\partial y + \gamma\partial/\partial z$, where α, β, γ are the usual unit vectors of three mutually perpendicular directions. However, this form of this very important differential operator is not at all a necessary form. In fact, if α and β are any two perpendicular unit vectors in a plane, and dr, ds are the corresponding differential distances in these two directions, then we have

$$\nabla = \alpha\partial/\partial r + \beta\partial/\partial s.$$

For instance, if functions are given in terms of r, θ, the usual polar coordinates, then $\nabla = U\rho\partial/\partial r + kU\rho\partial/r\partial\theta$. The proof that for any orthogonal set of curves a similar form is possible, is left to the student. In general, ∇ is defined as follows: ∇ is a linear differentiating vector operator connected with the variable vector ρ as follows: Consider first, a scalar function of ρ, say $F(\rho)$. Differentiate this by giving ρ any arbitrary differential $d\rho$. The result is linear in $d\rho$, and may be looked upon as the product of the length of $d\rho$ and the projection upon the direction of $d\rho$ of a certain vector for each direction $d\rho$. If now these vectors so projected can be reduced to a single vector, this is by definition ∇F. For instance, if F is the distance from the origin, then the differential of F in any direction is the projection of dr in a radial direction upon the direction of differentiation. Hence, $\nabla T\rho = U\rho$. In the case of plane vectors, ∇F will lie in the plane. In case the differential of F is polydromic, we define ∇F as a polydromic vector, which amounts to saying that a given set of vectors will each furnish its own differential value of dF. In some particular regions, or at certain points, the value of ∇F may become indefinite in direction because the differentials in all directions vanish. Of course, functions can be defined which would require careful investigation as to their differentiability, but we shall not be concerned with such in this work, and for their adequate treatment reference is made to the standard works on analysis.

We must consider next the meaning of ∇ as applied to vectors. It is evident that if ∇ is to be a linear and therefore distributive operator, then such an expression as $\nabla\sigma$ must have the same meaning as $\nabla X\alpha + \nabla Y\beta + \nabla Z\gamma$ if $\sigma = X\alpha + Y\beta + Z\gamma$, where α, β, γ are any independent constant vectors. This serves then as the definition of

$\nabla\sigma$, the only remaining necessary part of the definition is the vector part which defines the product of two vectors. This will be considered as we proceed.

15. **Nabla as a Complex Number.** We will consider now ρ to represent the complex number $x + yk$, or r_θ, and that all our expressions are complex numbers. The proper expression for ∇ becomes then

$$\nabla = \partial/\partial x + k\partial/\partial y = U\rho\partial/\partial r + kU\rho\partial/r\partial\theta.$$

In general for the plane, let ρ depend upon two parameters u, v, and let

$$d\rho = \rho_1 du + \rho_2 dv.$$

If σ is a function of ρ (generally not analytic in the usual sense) and thus dependent on u, v, we will have

$$d\sigma = \partial\sigma/\partial u\cdot du + \partial\sigma/\partial v\cdot dv = R\cdot d\rho\overline{\nabla}\cdot\sigma.$$

If we multiply $d\rho$ by $k\bar{\rho}_1$, which is perpendicular to $\bar{\rho}_1$, the real part of both sides will be equal and we have, since $k\rho_1$ is perpendicular to ρ_1,

$$Rk\bar{\rho}_1 d\rho = dvRk\bar{\rho}_1\rho_2,$$

and similarly

$$Rk\bar{\rho}_2 d\rho = duRk\bar{\rho}_2\rho_1 = -\, duRk\bar{\rho}_1\rho_2$$

since the imaginary part of $\bar{\rho}_1\rho_2$ equals $-$ the imaginary part of $\bar{\rho}_2\rho_1$.

Substituting in $d\sigma$ we have

$$d\sigma = R\cdot d\rho\left(-\frac{k\bar{\rho}_2}{Rk\bar{\rho}_1\rho_2}\frac{\partial}{\partial u} + \frac{k\bar{\rho}_1}{Rk\bar{\rho}_1\rho_2}\frac{\partial}{\partial v}\right)\cdot\sigma.$$

The expression in (), however, is exactly what we have defined above as $\overline{\nabla}$, and thus we have proved that we may write ∇ in the form corresponding to $d\rho$ in terms of u and v:

$$\nabla = k(\rho_2\partial/\partial u - \rho_1\partial/\partial v)/Rk\bar{\rho}_1\rho_2.$$

In case ρ_1 and ρ_2 are perpendicular the divisor evidently

reduces to $\pm\ T\rho_1 T\rho_2$ according as ρ_2 is negatively perpendicular to ρ_1 or positively perpendicular to it. We may write ∇ in this case in the form (since $\rho_2 = -k\rho_1 \cdot T\rho_2/T\rho_1$ or $+k\rho_1 \cdot T\rho_2/T\rho_1$)

$$\nabla = \frac{\rho_1}{T\rho_1^2}\frac{\partial}{\partial u} + \frac{\rho_2}{T\rho_2^2}\frac{\partial}{\partial v} = \bar{\rho}_1^{-1}\frac{\partial}{\partial u} + \bar{\rho}_2^{-1}\frac{\partial}{\partial v}.$$

In any case we have $dF = Rd\rho\overline{\nabla}F$, $d\sigma = Rd\rho\overline{\nabla}\cdot\sigma$.

Also in any case $\nabla = \nabla u \cdot \partial/\partial u + \nabla v \cdot \partial/\partial v$.

16. **Curl, Divergence, and Nabla.** Suppose now that α is the complex number for the unit tangent of one of a set of vector lines, and β the complex number for the unit tangent of the orthogonal set, at the same point. The curvature of the orthogonal set is the intensity of the vector rate of change of β along the orthogonal curve. But this is the same as the rate of change of the unit tangent α as we pass along the orthogonal curve from one vector line to an adjacent one. The differential of α is perpendicular to α, and hence parallel to the direction of β. Hence this curvature can be written

$$R\cdot\beta(R\beta\overline{\nabla}\cdot\alpha).$$

But if we also consider the value of $R\cdot\bar{\alpha}(R\cdot\alpha\overline{\nabla})\alpha$, since the differential of α in the direction of α has no component parallel to α, this term is zero, and may be added to the preceding without affecting its value. Hence the curvature of the orthogonal set reduces to

$$R(\bar{\alpha}R\alpha\overline{\nabla} + \beta R\beta\overline{\nabla})\alpha = R\cdot\overline{\nabla}\alpha.$$

This is the divergence of the curves of α.

If now $\sigma = T\sigma\cdot\alpha$, we find from the definition of the divergence of σ that it is merely

$$R\cdot\overline{\nabla}\sigma.$$

Considering in the same manner the definition of curl of σ,

we find it reduces to $-R\cdot k\overline{\nabla}\sigma$, and if we multiply this by k, so that we have

$$\text{curl } \sigma = -kRk\overline{\nabla}\sigma = I\cdot\overline{\nabla}\sigma,$$

we see at once that when added to the expression for the divergence of σ we have

$$\text{div}\cdot\sigma + \text{curl } \sigma = \overline{\nabla}\sigma.$$

The real part of this expression is therefore the divergence of σ, and the imaginary part is the curl of σ. This will agree with expressions for curl and divergence for space of three dimensions. We have thus found some of the remarkable properties of the operator ∇.

17. **Solenoidal and Lamellar Vector Fields.** When the divergence of σ is everywhere zero, the field is said to be *solenoidal.* If the curl is everywhere zero, the field is called *lamellar.*

18. **Properties of the Field.** Let a set of curves $u = c$ be considered, and the orthogonal set $v = a$, and let the field σ be expressed in the form

$$\sigma = X\nabla u + Y\nabla v,$$

where it is assumed that the gradients ∇u, ∇v exist at all points to be considered. We have then

$$\begin{aligned}\text{div}\cdot\sigma = R\overline{\nabla}\sigma = R\overline{\nabla}X\nabla u + R\overline{\nabla}Y\nabla v \\ + XR\overline{\nabla}\nabla u + YR\overline{\nabla}\nabla v.\end{aligned}$$

The expression $R\overline{\nabla}\nabla u$ is called the *plane dissipation of u.* In case it vanishes it is evident that u satisfies Laplace's equation, and is therefore *harmonic.*

We also have

$$\text{curl } \sigma = I\cdot\overline{\nabla}\sigma = -kRk\overline{\nabla}X\nabla u - kRk\overline{\nabla}Y\nabla v,$$

the other parts vanishing.

Since we have chosen orthogonal sets of curves we may write these in the forms

$$\operatorname{div}\cdot\sigma = (T\nabla u)^2\partial X/\partial u + (T\nabla v)^2\partial Y/\partial v + XR\overline{\nabla}\nabla u + YR\overline{\nabla}\nabla v,$$

$$\operatorname{curl}\sigma = (T\nabla u)(T\nabla v)(\partial Y/\partial u - \partial X/\partial v)k.$$

In case we have chosen the lines of σ for the u curves, then $X = 0$, and $\sigma = Y \nabla v$

$$\operatorname{div}\cdot\sigma = YR\overline{\nabla}\nabla v + (T\nabla v)^2\partial Y/\partial v,$$
$$\operatorname{curl}\sigma = T\nabla u T\nabla v\, \partial Y/\partial u\cdot k.$$

We notice that curl $\nabla u = 0$, curl $\nabla v = 0$, $\operatorname{div}\cdot k\nabla u = 0$, $\operatorname{div}\cdot k\nabla v = 0$, $k\nabla u = \nabla v T\nabla u/T\nabla v$, and for

$$Y = T\nabla u/T\nabla v,$$

we have

$$(T\nabla u)^{-2}R\overline{\nabla}\nabla u = \partial \log (T\nabla u/T\nabla v)/\partial u,$$
$$(T\nabla v)^{-2}R\overline{\nabla}\nabla v = \partial \log (T\nabla v/T\nabla u)/\partial v.$$

We may now draw some conclusions as to the types of curves and σ. (Cf. B. O. Peirce, *Proc. Amer. Acad. Arts and Sci.*, 38 (1903) 663–678; 39 (1903) 295–304.)

(1) The field will be solenoidal if $\operatorname{div}\cdot\sigma = 0$, hence

$$\partial \log Y/\partial v = -R\overline{\nabla}\nabla v/T\nabla v^2,$$

which may be integrated, giving

$$Y = e^{f(u,\ v)\ +\ g(u)}.$$

If v is harmonic, Y is a function of u only and $\sigma = G(u)\nabla v$.

(2) If the field is lamellar, curl $\sigma = 0$, and Y is a function of v only, so that $\sigma = H(v)\nabla v = \nabla L(v)$.

(3) If the field is both solenoidal and lamellar,

$$R\overline{\nabla}\nabla L(v) = 0, \quad \text{whence} \quad R\nabla\overline{\nabla}v/(T\nabla v)^2 = f(v),$$

which is a condition on the character of the curves. Hence

it is not possible to have a solenoidal and lamellar field with purely arbitrary curves.

(4) If the field is solenoidal and $T\sigma$, the intensity, is a function of u alone, $Y = p(u)/T\nabla v$, and therefore $\partial \log Y/\partial v = -\partial T\nabla v/T\nabla v\partial v = -R\overline{\nabla}\nabla v/T\nabla v^2$, whence

$$2R\overline{\nabla}\nabla v = \partial(T\nabla v)^2/\partial v,$$

which is a condition on the curves. An example is the cross-section of a field of magnetic intensity inside an infinitely long cylinder of revolution which carries lengthwise a steady current of electricity of uniform current density.

(5) If σ is lamellar and $T\sigma$ is a function of v only, $T\nabla v = g(v)$. An example is the field of attraction within a homogeneous, infinitely long cylinder of revolution. The condition is a restriction on the possible curves.

(6) If the field is lamellar and $T\sigma$ a function of u only, since Y is a function of v only, $\partial \log T\nabla v/\partial u = k(u)$, or $T\nabla v = l(u)/m(v)$.

This restricts the curves.

(7) If the field is solenoidal and $T\sigma$ a function of v only, $T\sigma = p(v)T\nabla v$. Therefore $d \log T\sigma/dv = \partial \log T\nabla\sigma/\partial v - R\overline{\nabla}\nabla v/(T\nabla v)^2$. Hence either both sides are constant or else both expressible in terms of v. If the field is not lamellar also, $T\nabla v$ must then be a function of u as well as of v.

(8) If the field is lamellar and has a scalar potential function, that is, $\sigma = \nabla P$, then since $\sigma = q(v)\nabla v$, we must have P a function of v only, and $\sigma = P'\nabla v$. From this it follows that $\operatorname{div}\cdot\sigma = P'(v)R\overline{\nabla}\nabla v + P''(v)(T\nabla v)^2$.

(9) If the field is uniform, $T\sigma = a$, $Y = a/T\nabla v$, and σ is lamellar only if $T\nabla v$ is either constant or a function of v only, while σ is solenoidal only if we have

$$2R\overline{\nabla}\nabla v = \partial(T\nabla v)^2/\partial v.$$

(10) Whatever function u is, the u lines are vector lines for the vectors $\xi = f(u)U\nabla v$, $\zeta = g(v)U\nabla v$, or

$$\eta = h(u, v)U\nabla v.$$

(11) If the field is solenoidal, $T\sigma$ a function of u only, and the u curves are the lines of the field, then the curl takes the form $-k\,\mathrm{div}\cdot k\sigma$, whence it has the form

$$k[b(u)R\overline{\nabla}\nabla u + b'(u)(T\nabla u)^2],$$

where b may be any differentiable function. If $T\sigma$ is also a function of v, the form of the curl is

$$k[b(u, v)R\overline{\nabla}\nabla u + \partial b(u, v)/\partial u(T\nabla u)^2].$$

(12) If $T\sigma$ is a function of u only, the divergence takes the form

$$\mathrm{div}\cdot\sigma = T\sigma[R\overline{\nabla}\nabla v/T\nabla v - \partial T\nabla v/\partial v].$$

(13) If $T\sigma$ is a function of v only

$$\mathrm{curl}\ \sigma = -kT\sigma T\nabla u/T\nabla v\cdot\partial T\nabla v/\partial u.$$

19. **Continuous Media.** When the field is that of the velocity of a continuous medium, we have two cases to take into account. If the medium is incompressible it is called a *liquid*, otherwise a *gas*. Incompressibility means that the density at a point remains invariable, and if this is c, then from

$$dc/dt = \partial c/\partial t + R\bar{\sigma}\nabla c, = \partial c/\partial t + R\overline{\nabla}(c\sigma) - cR\overline{\nabla}\sigma$$

we see that the first two terms together vanish, giving the *equation of continuity*, since they give the rate per square centimeter at which actual material (density times area, since the height is constant) is changing. Hence in this case

$$dc/dt = -cR\overline{\nabla}\sigma.$$

This gives the rate of change of the density at a point moving with the fluid. Hence if it is incompressible, the velocity is solenoidal, $R\overline{\nabla}\sigma = 0$.

This may also be written curl $(-k\sigma) = 0$, hence $-k\sigma = \nabla Q$, and $\sigma = k\nabla Q$, which shows that for every liquid there is a function Q called the *function of flow.*

When curl $\xi = 0$, we have seen that ξ is called lamellar. It may also be called *irrotational,* since the curl is twice the angular rate of rotation of the infinitesimal parts of the medium, about axes perpendicular to the plane, and if curl $\xi = 0$ there is no such rotation. Curl is analogous to density, being a density of rotation when the vector field is a velocity field.

The *circulation* of the field is the integral $\int R\bar{\sigma}d\rho$ along any path from a point A to a point B. This is the same as $Xdx + Ydy$, and is exact when

$$\partial X/\partial y = \partial Y/\partial x.$$

But this gives exactly the condition that the curl should vanish. Hence if the motion is irrotational the circulation from one point to another is independent of the path. In this case we may write $\sigma = \nabla P$ where P is called the *velocity potential.*

When σ is irrotational, the lines of Q have as orthogonals the lines of P. If the motion is rotational, these orthogonals are not the lines of such a function as P. If the motion is irrotational, we have for a liquid, $R\bar{\nabla}\nabla P = 0$, and P must be harmonic. Hence if the orthogonal curves of the Q curves can belong to a harmonic function they can be curves of a velocity potential. If a set of curves belong to the harmonic function u, then $R\bar{\nabla}\nabla u = 0$, and this shows that the curl of $-k\nabla u$ is zero, whence $Rd\bar{\rho}(-k\nabla u)$ is exact $= dv$, where $\nabla v = -k\nabla u$. From this we have $\nabla u = k\nabla Q$ for the condition that the orthogonal curves belong to a harmonic function. This however gives the equation $T\nabla u = T\nabla Q$. We may assert then for a liquid that there is always a function of flow, and the curves belonging to

this function are the vector lines of the velocity, the intensity of the velocity being the intensity of the gradient of the function of flow. If the orthogonal curves belong to a function which has a gradient of the same intensity, both functions are harmonic, the function of the orthogonal set is a velocity potential, and the motion is irrotational.

We have a simple means of discovering the sets of curves that belong to harmonic functions, as is well known to students of the theory of functions of a complex variable, since the real and the imaginary part of an analytic function of a complex variable are harmonic for the variable coordinates of the variable. That is to say, if $\rho = x + yk$, and $\xi = f(\rho) = u + vk$, then u, v are harmonic for x, y. The condition given by Cauchy amounts to the equation $\nabla u = -k\nabla v$, or $\nabla\xi = 0$ where ξ is a complex number. It is clear from this that the field of $\bar{\xi}$ is both solenoidal and lamellar, a necessary and sufficient condition that ξ be an analytic function of a complex variable. In this case ξ is called a *monogenic* function of position in the plane. It is clear that $\xi = \overline{\nabla} H$ where H is a harmonic function.

In case there are singularities in the field it is necessary to determine their effect on the integrals. For instance, if we have a field σ and select a path in it, from A to B, or a loop, the flux of σ through the path is the integral of the projection of σ on the normal of the path, that is, if the path is a curve given by $d\rho$, so that the projection is $R\bar{\sigma}(-kd\rho)$, the integral of this is the flux through the path. It is written

$$\Sigma = \int_A^B (-R\bar{\sigma}kd\rho) = -k\int I\bar{\sigma}d\rho.$$

In the case of a liquid the condition $R\overline{\nabla}\sigma = 0$ shows that the expression is integrable over any path from A to B, with the same value, unless the two paths enclose a singularity of the field. In the case of a node, the integral around

a loop enclosing the node is called the *strength of the source or sink* at the node. We may imagine a constant supply of the liquid to enter the plane or to leave it at the node, and be moving along the lines of the field. Such a system was called by Clifford a *squirt.*

If the circulation is taken around a singular point it will usually have a different value for every turn around the point, giving a polydromic function. These peculiarities must be studied carefully in each case.

EXERCISES

1. From $\xi = A\rho^n$ we find in polar coordinates that

$$u = Ar^n \cos n\theta, \qquad v = Ar^n \sin n\theta.$$

These functions are harmonic and their curves orthogonal. Hence if we set $\sigma = \nabla u$ or $\sigma = \nabla v$, we shall have as the vector lines of σ the v curves or the u curves. What are the curves for the cases $n = -3, -2, -1, 1, 2, 3$? What are the singularities?

2. Study $\xi = A \log \rho$, and $\xi = A \log (\rho - \alpha)/(\rho + \alpha)$.

3. Consider the function given implicitly by $\rho = \xi + e^{\xi}$. This represents the flow of a liquid into or out of a narrow channel, in the sense that it gives the lines of flow when it is not rotational.

4. Show that $\sigma = A/\bar{\rho}$ gives a radial irrotational flow, while $\sigma = Ak/\bar{\rho}$ gives a circular irrotational flow. What is true of $\sigma = Ak\rho$? The last is Clifford's *Whirl.*

5. Study a flow from a source at a given point of constant strength to a sink at another point, of the same strength as the source.

6. If the lines are concentric circles, and the angular velocity of any particle about the center is proportional to the n-th power of the radius of the path of the point, show that the curl is $\frac{1}{2}(n + 2)$ times the angular velocity.

7. A point in a gas is surrounded by a small loop. Show that the average tangential velocity on the loop has a ratio to the average normal velocity which is the ratio of the tensor of the curl to the divergence.

8. What is the velocity when there is a source at a fixed origin, and the divergence varies inversely as the n-th power of the distance from the origin. [The velocity potential is $A \log r - B(n - 2)^{-2}r^{2-n}$.]

9. Consider the field of two sources of equal strength. The lines are for irrotational motion, cassinian ovals, where, if r, r' are the distances

from the two sources (foci) and $rr' = h^2$, $Q = A \log h + B$, the velocity is such that $T\sigma = AT\rho/h^2$, the origin being half way between the foci; the orthogonal curves are given by $u = \frac{1}{2}A[\pi/2 - (\theta + \theta_1)]$ where θ, θ_1 are the angles between the axis and the radii from the foci, that is they are equilateral hyperbolas through the foci. The circulation about one focus is πA, about both $2\pi A$.

10. If the lines are confocal ellipses given by

$$x^2/\mu + y^2/(\mu - c^2) = 1,$$

then $Q = A \log (\sqrt{\mu} + \sqrt{(\mu - c^2)}) + B$. If ρ is the perpendicular from the center upon the tangent of the ellipse at any point, then the velocity at the point is such that $T\sigma = -Ap/\sqrt{[\mu(\mu - c^2)]}$, and the direction of σ is the unit normal. The potential function is $A \sin^{-1} B' \sqrt{\nu}/c$. $\sqrt{\nu}$ is the semi-major axis. What happens at the foci?

11. If the stream lines are the hyperbolas of the preceding, then $\sigma = 2A\sqrt{(\nu/(\mu - \nu))}$ times the unit normal of the hyperbola. On the line $\rho = yk\alpha$ there is no velocity, at the foci the velocity is ∞, half way between it is 0. The lines along the major axis outside the foci act like walls.

12. If we write for brevity u_1 for $T\nabla u$, and v_1 for $T\nabla v$, show that we have whether the u curves are orthogonal to the v curves or not,

$$\overline{\nabla}\nabla = u_1^2\partial^2/\partial u^2 + v_1^2\partial^2/\partial v^2 + \overline{\nabla}\nabla u\partial/\partial u + \overline{\nabla}\nabla v\partial/\partial v + 2R\overline{\nabla u}\nabla v\partial^2/\partial u\partial v.$$

If the sets of curves are orthogonal the last term vanishes; if u and v are harmonic the third and fourth terms drop out; if both cases happen, only the first two terms are left.

13. In case of polar coordinates, $\nabla r = U\rho$, $\nabla\theta = r^{-2}k\rho$ and

$$\overline{\nabla}\nabla = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\theta^2.$$

14. A gas moves in a plane in lines radiating from the origin, which is a source. The divergence is a function of r only, the distance from the center. Find the velocity and the density at any point.

$$\sigma = \rho f(r), \qquad R\overline{\nabla}\sigma = e(r) = 2f(r) + rf'(r),$$

and

$$f(r) = A/r^2 + r^{-2}\int re(r)dr.$$

To determine c,

$$R\overline{\nabla} \log c\sigma = -e(r) = f(r)R\rho\nabla \log c = rf(r)\partial \log c/\partial r.$$

15. Show that in the steady flow of a gas we may find an integrating factor for $Rd\bar{\rho}k\sigma$ by using the density. [$\partial c/\partial t = 0 = R\overline{\nabla}c\sigma = \text{curl}\cdot kc\sigma$, and $Rd\bar{\rho}kc\sigma$ is exact.]

16. A fluid is in steady motion, the lines being concentric circles. The curl is known at each point and the tensor of σ is a function of r only. Find the velocity and the divergence.

17. Rotational motion, that is a field which is not lamellar, is also called *vortical* motion. The points at which the curl does not vanish may be distributed in a continuous or a discontinuous manner. In fact there may be only a finite number of them, called vortices. We have the following:

$$\sigma = k\nabla Q, \qquad \overline{\nabla}\nabla Q = T \text{ curl } \sigma = 2\omega,$$
$$Q = \pi^{-1} \int \int \omega' \log r dx'dy' + Q_0,$$

where ω' denotes ω at the variable point of the integration, r is the variable distance from the point at which the velocity is wanted, and Q_0 is any solution of Laplace's equation which satisfies the boundary conditions.

If the mass is unlimited and is stationary at infinity we have

$$\sigma = k/\pi \int \int \omega'(\rho - \rho')/T(\rho - \rho')^2 \cdot dx'dy'.$$

A single vortex filament at ρ' of strength l would give the velocity

$$\sigma = l/2\pi \cdot (\rho - \rho')/T(\rho - \rho')^2.$$

If we multiply the velocity at each point ρ at which there is a vortex by the strength, and integrate over the whole field, we find the sum is zero. There is then a center of vortices where the velocity is zero, something like a center of gravity. Instances are

(1) A single vortex of strength l. The vortex point will remain at rest, and points distant from it r will move on concentric circles with the vortex as center, and velocity $l/2\pi r$. The circulation of any loop surrounding the vortex is of course the strength.

(2) Two vortices of strengths l_1, l_2. They will rotate about the common center of gravity of two weighted points at the fixed distance apart a, the weights being the two strengths. The angular velocity of each is

$$\frac{l_1 + l_2}{2\pi a^2}.$$

The stream lines of the field are given by $r_1{}^{l_1} r_2{}^{l_2} = \text{const.}$ When $l_1 = -l_2$ the center is at infinity, and the vortices remain a fixed distance apart, moving parallel to the perpendicular bisector of this segment joining them. Such a combination is called a vortex pair. The stream lines of the accompanying velocity are coaxal circles referred to the moving points as limit points. The plane of symmetry may be taken as a boundary since it is one of the stream lines, giving the motion of a single vortex in a field bounded by a plane, the linear velocity of the vortex being parallel to the wall and $\frac{1}{4}$ of the velocity of the liquid along the wall. The figure suggests the method of images which can indeed be applied. For further problems of the same character works on Hydrodynamics should be consulted.

18. Liquid flows over an infinite plane towards a circular spot where it leaks out at the rate of 2 cc. per second for each cm.[2] area of the leaky portion. The liquid has a uniform depth of 10 cm. over the entire plane field. Find formulas for the velocity of the liquid inside the region of the leaky spot, and the region outside, and show that there is a potential in both regions.

$\sigma = \frac{1}{10}\rho$ in spot, $40/\bar{\rho}$ outside, $P = \frac{1}{20}\rho\rho$ in spot, $40 \log T\rho - 20 \log 400$ outside.

Find the flux through a plane area 20 cm. long and 10 cm. high, whose middle line is 5 cm. from the center of the leaky spot, also when it is 30 cm. from the leaky spot. Find the divergence in the two regions.
Franklin, *Electric Waves*, pp. 307–8.

19. Show that in an irrotational motion with sources and sinks, the lines of flow are the orthogonal curves of the stream lines of a corresponding field in which the sources and the sinks are replaced by vortices of strengths the same as that of the sources and sinks, and inversely. Stream lines and levels change place as to their roles. For sources and sinks $Q = 1/2\pi \cdot \Sigma l_1\theta_1$, $P = 1/2\pi \cdot \Sigma \log r_1{}^{l_1}$.

20. Vector Potential. In the expression $\sigma = -\nabla kQ$ we express σ as a vector derived by the operation of ∇ upon $-kQ$, the latter being a complex number. In such a case we may extend our terminology and call $-kQ$ the *vector potential* of σ. A vector may be derived from more than one vector potential. In order that there be a vector potential it is necessary and sufficient that the divergence of σ vanish. Hence any liquid flow can have a vector potential, which is indeed the current function multiplied by $-k$. It is clear that Q must be harmonic.

CHAPTER VI

VECTORS IN SPACE

1. **Biradials.** We have seen that in a plane the figure made up of two directed segments from a vertex enables us to define the ratio of the two vectors which constitute the sides when the figure is in some definite position. This ratio is common to all the figures produced by rotating the figure about a normal of the plane through its vertex, and translating it anywhere in the plane. We may also reduce the sides proportionately and still have the same ratio. The ratio is a complex number or, as we will say in general, a hypernumber.

If now we consider vectors in space of three dimensions, we may define in precisely the same manner a set of hypernumbers which are the ratios of the figures we can produce in an analogous manner. Such figures will be called *biradials*. To each biradial there will correspond a hypernumber. Besides the translation and the rotation in the plane of the two sides of the biradial, we shall also permit the figure to be transferred to any parallel plane. This amounts to saying that we may choose a fixed origin, and whatever vectors we consider in space, we may draw from the origin two vectors parallel and equal to the two considered, thus forming a biradial with the origin as vertex. Then any such biradial will determine a single hypernumber. Further the hypernumbers which belong to the biradials which can be produced from the given biradial by rotating it in its plane about the vertex will be considered as equal.

The hypernumbers thus defined are extensions of those we have been using in the preceding chapter, the new feature being the different hypernumbers k which we now need, one new k in fact for each different plane through the given vertex. This gives us then a double infinity of hypernumbers of the complex type, $r \cdot \text{cks}\ \theta$, where the double infinity of k's constitute the new elements.

2. **Quaternions.** The hypernumbers we have thus defined metrico-geometrically involve four essential parameters in whatever way they are expressed, since the biradials involve two and the plane in which they lie two more. Hence they were named by Hamilton *Quaternions.* In order to arrive at a fuller understanding of their properties and relations, we will study the geometric properties of biradials.

In the first place if we consider any given biradial, there is involved in its quaternion, just as for the complex number in the preceding chapter, two parts, a real part and an imaginary part, and we can write the quaternion in the form

$$q = r \cos \theta + r \sin \theta \cdot \alpha,$$

where α corresponds to what was written k in the preceding chapter, and is a hypernumber determined solely by the plane of the biradial. On account of this we may properly *represent* α by a unit normal to the plane of the biradial, so taken that if the angle of the biradial is considered to be positive, the direction of the normal is such that a right-handed screw motion turning the initial vector of the biradial into the terminal vector in direction would involve an advance along the normal in the direction in which it points. It is to be understood very clearly that the unit vector α and the hypernumber α are distinct entities, one merely representing the other. The real

part of q is called, according to Hamilton's terminology, the *scalar* part of q, and written Sq. The imaginary part is called, on account of the representation of α as a vector, the *vector* part of q and written Vq. The unit α is called the *unit vector* of q and written UVq. The angle of q is θ and written $\angle q$. The number r which is the ratio of the lengths of the sides of the biradial is called the *tensor* of q, and written Tq. The expression $\cos\theta + \sin\theta \cdot \alpha = cas \cdot \theta$ is called the *versor* of q, and written Uq.

Sq is a quaternion for which $\theta = 0°$ or $180°$, Vq is a quaternion for which $\theta = 90°$ or $270°$. Tq is a quaternion of $0°$, being always positive. α is a quaternion of $\theta = 90°$, and sometimes called a *right versor*.

3. **Sum of Quaternions.** In order to define the sum of two quaternions we define the sum of two biradials first. This is accomplished by rotating the two biradials in their planes until their initial lines coincide, and then diminishing or magnifying the sides of one until the initial vectors are exactly equal and coincide. This is always possible. We then define as the sum of the two biradials, the biradial whose initial vector is the common vector of the two, and terminal vector is the vector sum of the two terminal vectors. The sum of the corresponding quaternions is then the quaternion of the biradial sum. Since vector addition is commutative, the addition of quaternions is commutative.

Passing now to the scalar and vector parts of the quaternions, we will prove that they can be added separately, the scalar parts like any numbers and the vector parts like vectors.

In the figure let the biradial of q be OB/OA, of r be OC/OA, and of $q + r$ be OD/OA. Let the vector part of q, $Tq \cdot \sin \angle q \cdot UVq$ be laid off as a vector Vq perpendicular

to the plane of the biradial of q, and similarly for Vr. Then we are to show that $V(q + r) = Vq + Vr$ in the representation and that this represents the vector part of $q + r$ according to the definition. It is evident that

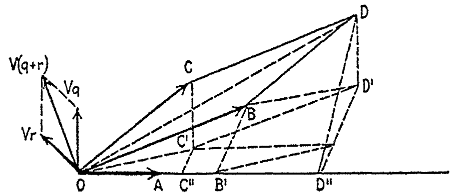

FIG. 12.

$OB = OB' + B'B$, the first vector along OA, the second perpendicular to OA. Also $OC = OC'' + C''C' + C'C$, the first part along OA, the second parallel to $B'B$, and the third perpendicular to the plane of OAB. The sum $OB + OC = OD$, where $OD = OD'' + D''D' + D'D$, and $OD'' = OB' + OC''$, $D''D' = B'B + C''C'$, $D'D = C'C$.

Hence the biradial of the sum is OD/OA, where the scalar part is the ratio of OD'' to OA. This is clearly the sum of the scalar parts of q and r, and

$$S(q + r) = Sq + Sr.$$

The vector part of the quaternion for OD/OA is the ratio of $D''D$ to OA in magnitude, and the unit part is represented by a unit normal perpendicular to OD'' and $D''D$. But $D''D = B'B + C''C$, and the ratio of $D''D$ to OA equals the sum of the ratios of $B'B$ and $C''C$ to OA. If then we draw, in a plane through O which is perpendicular to OA, the vector Vq along the representative unit normal of the plane OAB, and of a length to represent the numerical ratio of $B'B$ to OA, and likewise Vr to represent the ratio of $C''C$ to OA laid off along the representative unit normal

to the plane OAC, because $D''D$ is parallel to this plane, as well as $B'B$ and $C''C$, the representative unit vector of $q + r$ will lie in the plane, and will be in length the vector sum of Vq and Vr, that is $V(q + r)$ as shown.

It follows at once since the addition of scalars is associative, and the addition of vectors is associative, and the two parts of a quaternion have no necessary precedence, that the addition of quaternions is associative.

4. **Product of Quaternions.** To define the product of quaternions we likewise utilize the biradials. In this case however we bring the initial vector of the multiplier to coincide with the terminal line of the multiplicand, and define the product biradial as the biradial whose initial vector is the initial vector of the multiplicand, and the terminal vector is the terminal vector of the multiplier. In the figure, the product of the biradials OB/OA, and

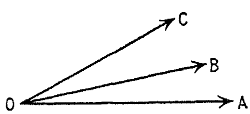

FIG. 13.

OC/OB, is, writing the multiplier first,

$$OC/OB \cdot OB/OA = OC/OA.$$

It is clear that the tensor of the product is the product of the tensors, so that

$$T \cdot qr = TqTr.$$

It follows that

$$U \cdot qr = UqUr.$$

It is evident from the figure that the angle of the product will be the face angle of the trihedral, AOC, or on a unit sphere would be represented by the side of the spherical

triangle corresponding. It is clear too that the reversal of the order of the multiplication will change the plane of the product biradial, usually, and therefore will give a quaternion with a different unit vector, though all the other numbers dependent upon the product will remain the same. However we can prove that multiplication of quaternions is associative. In this proof we may leave out the tensors and handle only the versors. The proof is due to Hamilton.

To represent the biradials, since the vectors are all taken as unit vectors, we draw only an arc on the unit sphere, from one point to the other, of the two ends of the two unit vectors of the biradial. Thus we represent the biradial of q by CA, or, since the biradial may be rotated in its plane about the vertex, equally by ED. The others involved are shown. The product qr is represented by FD, from the definition, or equally by LM. What we have to prove is that the product $p \cdot qr$ is the same as the product $pq \cdot r$, that is, we must prove that the arcs KG and LN are on the same great circle and of equal length and direction.

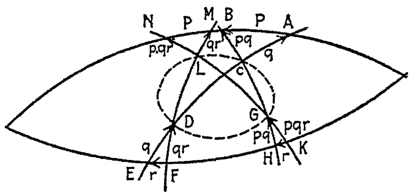

FIG. 14.

Since $FE = KH$, $ED = CA$, $HG = CB$, $LM = FD$, the points L, C, G, D are on a spherical conic, whose cyclic planes are those of AB, FE, and hence KG passes through L, and with LM intercepts on AB an arc equal to AB. That is, it passes through N, or KG and LN are arcs of the

same great circle, and they are equal, for G and L are points in the spherical conic.

5. **Trirectangular Biradials.** A particular pair of biradials which lead to an interesting product is a pair of which the vectors of each biradial are perpendicular unit vectors, and the initial vector of one is the terminal of the other, for in such case, the product is a biradial of the same kind. In fact the three lines of the three biradials form a trirectangular trihedral. If the quaternions of the three

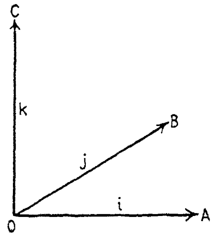

FIG. 15.

are i, j, k, then we see easily that the quaternion of the biradial OC/OB is represented completely by the unit vector marked i, the quaternion of OA/OC by j, and of OB/OA by k. The products are very interesting, for we have

$$ij = k, \qquad jk = i, \qquad ki = j,$$

and if we place the equal biradials in the figure we also have

$$ji = -k, \qquad kj = -i, \qquad ik = -j.$$

Furthermore, we also can see easily that, utilizing the common notation of powers,

$$i^2 = -1, \qquad j^2 = -1, \qquad k^2 = -1.$$

Since it is evidently possible to resolve the vector part of any quaternion, when it is laid off on the unit vector of its plane as a length, into three components along the directions of i, j, k, and since the sum of the vector parts of

quaternions has been shown to be the vector part of the sum, it follows that any quaternion can be resolved into the parts

$$q = w + xi + yj + zk.$$

These hypernumbers can easily be made the base of the whole system of quaternions, and it is one of the many methods of deriving them. Hamilton started from these. The account of his invention is contained in a letter to a friend, which should be consulted. (*Philosophical Magazine*, 1844, vol. 104, ser. 3, vol. 25, p. 489.)

6. **Product of Vectors.** It becomes evident at once if we consider the product of two vector parts of quaternions, or two quaternions whose scalar parts are zero, that we may consider this product, a quaternion, as the product of the vector lines which represent the vector parts of the quaternion factors. From this point of view we ignore the biradials completely, and look upon every geometric vector as the representative of the vector part of a set of quaternions with different scalars, among which one has zero scalar. From the biradial definition we have

$$VqVr = S \cdot VqVr + V \cdot VqVr$$

equal to the quaternion whose biradial consists of two vectors in the same plane as the vector normals of the

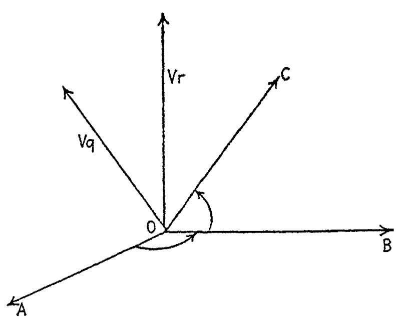

Fig. 16.

biradials of Vq, Vr and perpendicular to them respectively. In the figure the biradial of Vr is OAB, and of Vq is OBC, and of $VqVr$ is OAC. If then we represent the vectors by Greek letters whether meant to be considered as lines or as vector quaternions, $\alpha = Vq$, $\beta = Vr$, then the quaternion which is the product of $\alpha\beta$ has for its angle the angle between β and $\alpha + 180°$, and for its normal the direction OB. If we take $UV\alpha\beta$ in the opposite direction to OB, and of unit length, so as to be a positive normal for the biradial $\alpha\ \beta$ in that order, then we shall have, letting θ be the angle from α to β,

$$\alpha\beta = T\alpha T\beta(-\cos\theta + UV\alpha\beta \sin\theta).$$

We can write at once then the fundamental formulae

$$S\cdot\alpha\beta = -\ T\alpha T\beta \cos\theta, \qquad V\cdot\alpha\beta = T\alpha T\beta\cdot\sin\theta\cdot UV\alpha\beta.$$

From this form it is clear also that any quaternion can be expressed as the product of two vectors, the angle of the two being the supplement of that of the quaternion, the product of their lengths being the tensor of the quaternion, and their plane having the unit vector of the quaternion as positive normal.

If now we consider the two vectors α and β to be resolved in the forms

$$\alpha = ai + bj + ck, \qquad \beta = li + mj + nk,$$

where i, j, k have the significance of three mutually trirectangular unit vectors, as above, then since $T\alpha T\beta \cos\theta = al + bm + cn$, and since the vector $T\alpha T\beta \sin\theta\cdot UV\alpha\beta$ is

$$(bn - cm)i + (cl - an)j + (am - bl)k,$$

we have

$$\alpha\beta = -\ (al + bm + cn) + (bn - cm)i \\ + (cl - an)j + (am - bl)k.$$

But if we multiply out the two expressions for α and β distributively, the nine terms reduce to precisely these. Hence we have shown that the multiplication of vectors, and therefore of quaternions in general, is distributive when they are expressed in terms of these trirectangular systems. It is easy to see however that this leads at once to the general distributivity of all multiplications of sums.

7. **Laws of Quaternions.** We see then that the addition and multiplication of quaternions is associative, that addition is commutative, and that multiplication is distributive over addition. Multiplication is usually not commutative. We have yet to define division, but if we now consider a biradial as not being geometric but as being a quaternion quotient of two vectors, we find that β/α differs from $\alpha\beta$ only in having its scalar of opposite sign, and its tensor is $T\beta/T\alpha$ instead of $T\alpha T\beta$.

It is to be noticed that while we arrived at the hypernumbers called quaternions by the use of biradials, they could have been found some other way, and in fact were so first found by Hamilton, whose original papers should be consulted. Further the use of vectors as certain kinds of quaternions is exactly analogous, or may be considered to be an extension of, the method of using complex numbers instead of vectors in a plane. In the plane the vectors are the product of some unit vector chosen for all the plane, by the complex number. In space a vector is the product of a unit vector (which would have to be drawn in the fourth dimension to be a complete extension of the plane) by the hypernumber we call a vector. However, the use of the unit in the plane was seldom required, and likewise in space we need never refer to the unit 1, from which the vectors of space are derived. On the other hand, just as in the plane all complex numbers can be found as the ratios

of vectors in the plane in an infinity of ways, so all quaternions can be found as the ratios of vectors in space. All vectors are thus as quaternions the ratios of perpendicular vectors in space. And multiplication is always of vectors as quaternions and not as geometric entities. In the common vector systems other than Quaternions, the scalar part of the quaternion product, usually with the opposite sign, and the vector part of the quaternion product, are looked upon as products formed directly from geometric considerations. In such case the vector product is usually defined to be a vector in the geometric sense, perpendicular to the two given vectors. Therefore it is a function of the two vectors and is not a number or hypernumber at all. In these systems, the scalar is a common number, and of course the sum of a number and a geometric vector is an impossibility. It seems clear that the only defensible logical ground for these different investigations is that of the hypernumber.

It is to be noticed too that Quaternions is peculiarly applicable to space of three dimensions, because of the duality existing between planes and their normals. In a space of four dimensions, for instance, a plane, that is a linear extension dependent upon two parameters, has a similar figure of two dimensions as normal. Hence, corresponding to a biradial we should not have a vector. To reach the extension of quaternions it would be necessary to define triradials, and the hypernumbers corresponding to them. Quaternions however can be applied to four dimensional space in a different manner, and leads to a very simple geometric algebra for four-dimensional space. The products of quaternions however are in that case not sufficient to express all the necessary geometrical entities, and recourse must be had to other functions of quaternions.

In three-dimensional space, however, all the necessary expressions that arise in geometry or physics are easily found. And quaternions has the great advantage over other systems that it is associative, and that division is one of its processes. In fact it is the most complex system of numbers in which we always have from $P \cdot Q = 0$ the conclusion $P = 0$, or $Q = 0$.*

8. **Formulae.** It is clear that if we reverse the order of the product $\alpha\beta$ we have

$$\beta\alpha = S\alpha\beta - V\alpha\beta.$$

This is called the conjugate of the quaternion $\alpha\beta$, and written $K\cdot\alpha\beta$. We see that

$$SKq = Sq = KSq, \qquad VKq = -Vq = KVq.$$

Further, since

$$qr = SqSr + SqVr + SrVq + VqVr,$$

we have

$$K\cdot qr = SqSr - SqVr - SrVq + VrVq = KrKq.$$

From this important formula many others flow. We have at once

$$K\cdot q_1\cdots q_n = Kq_n\cdots Kq_1.$$

And for vectors

$$K\alpha_1\cdots\alpha_n = (-)^n\alpha_n\cdots\alpha_1.$$

Since

$$Sq = \tfrac{1}{2}(q + Kq), \qquad Vq = \tfrac{1}{2}(q - Kq),$$

we have therefore

$$S\cdot\alpha_1\cdots\alpha_{2n} = \tfrac{1}{2}(\alpha_1\cdots\alpha_{2n} + \alpha_{2n}\cdots\alpha_1),$$

$$S\cdot\alpha_1\cdots\alpha_{2n-1} = \tfrac{1}{2}(\alpha_1\cdots\alpha_{2n-1} - \alpha_{2n-1}\cdots\alpha_1),$$

$$V\cdot\alpha_1\cdots\alpha_{2n} = \tfrac{1}{2}(\alpha_1\cdots\alpha_{2n} - \alpha_{2n}\cdots\alpha_1),$$

$$V\cdot\alpha_1\cdots\alpha_{2n-1} = \tfrac{1}{2}(\alpha_1\cdots\alpha_{2n-1} + \alpha_{2n-1}\cdots\alpha_1).$$

* Consult *Dickson:* Linear Algebras, p. 11.

In particular

$$2S\alpha\beta = \alpha\beta + \beta\alpha, \qquad 2S\alpha\beta\gamma = \alpha\beta\gamma - \gamma\beta\alpha,$$
$$2V\alpha\beta = \alpha\beta - \beta\alpha, \qquad 2V\alpha\beta\gamma = \alpha\beta\gamma + \gamma\beta\alpha.$$

It should be noted that these formulae show us that both the scalar and the vector parts of the product can themselves always be reduced to combinations of products.

This is simply a statement again of the fact that in quaternions we have only one kind of multiplication, which is distributive and associative.

We see from the expanded form above for $S\cdot qr$ that

$$S\cdot qr = S\cdot rq.$$

Hence, in any scalar part of a product, the factors may be permuted cyclically. For instance,

$$S\cdot\alpha\beta = S\cdot\beta\alpha, \qquad S\cdot\alpha\beta\gamma = S\cdot\beta\gamma\alpha = S\cdot\gamma\alpha\beta,$$
$$S\cdot\alpha\beta\gamma\delta = S\cdot\beta\gamma\delta\alpha = \cdots.$$

From the form of

$$Sq = \tfrac{1}{2}(q + Kq), \qquad Sq = SKq;$$

hence we have

$$S\alpha\beta = S\beta\alpha, \qquad S\alpha\beta\gamma = -\,S\gamma\beta\alpha, \qquad S\alpha\beta\gamma\delta = S\delta\gamma\beta\alpha, \text{ etc.}$$

From the form of $VKq = -\,Vq$ we see that

$$V\alpha\beta = -\,V\beta\alpha, \qquad V\alpha\beta\gamma = V\gamma\beta\alpha,$$
$$V\alpha\beta\gamma\delta = -\,V\delta\gamma\beta\alpha, \qquad V\alpha\beta\gamma\delta\epsilon = V\epsilon\delta\gamma\beta\alpha\cdots.$$

We do not have a simple relation between $V\cdot qr$ and $V\cdot rq$, but we have the fact that they are respectively the sum and the difference of two vectors, namely,

If $\alpha = SqVr + SrVq$, $\beta = VVqVr$, then β is perpendicular to α, and

$$Vqr = \alpha + \beta, \qquad Vrq = \alpha - \beta.$$

It is obvious that $TVqr = TVrq$ and that $\angle qr = \angle rq = \tan^{-1} TVqr/Sqr$. The planes differ.

The product of q and Kq is the square of the tensor of q. We indicate the unitary part of q, called the versor of q, by Uq. We have then the formulae

$$q = w + ix + jy + kz, \qquad Kq = w - ix - jy - kz,$$
$$(Tq)^2 = w^2 + x^2 + y^2 + z^2, \qquad Uq = \frac{w + ix + jy + kz}{Tq},$$

$$Sq = w, \qquad Vq = ix + jy + kz,$$
$$(TVq)^2 = x^2 + y^2 + z^2, \qquad UVq = \frac{ix + jy + kz}{TVq},$$

$$(TVUq)^2 = (x^2 + y^2 + z^2)/(w^2 + x^2 + y^2 + z^2),$$
$$\cos\cdot \angle\ q = w/Tq = S\cdot Uq,$$
$$\sin\cdot \angle\ q = TVq/Tq = TVUq,$$
$$\angle\cdot q = \tan^{-1} TVq/Sq.$$

The product of two quaternions is

$$qr = ww' - xx' - yy' - zz' + i(wx' + w'x + yz' - y'z) + j(wy' + w'y + zx' - z'x) + k(wz' + w'z + xy' - x'y).$$

From the formula $Tqr = TqTr$ we have a noted identity

$$(w^2 + x^2 + y^2 + z^2)(w'^2 + x'^2 + y'^2 + z'^2) = (ww' - xx' - yy' - zz')^2 + (wx' + w'x + yz' - y'z)^2 + (wy' + w'y + zx' - z'x)^2 + (wz' + w'z + xy' - x'y)^2.$$

This formula expresses the sum of four squares as the product of the sums of four squares. It was first given by Euler. The problem of expressing the sum of three squares as the product of sums of three or four squares and the sum of eight squares as the product of sums of eight squares has also been considered.

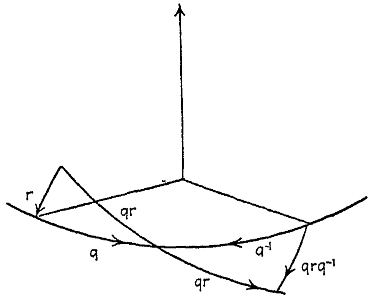

FIG. 17.

9. **Rotations.** We see from the adjacent figure that we have for the product

$$qrq^{-1}$$

a quaternion of tensor and angle the same as that of r. But the plane of the product is produced by rotating the plane of r about the axis of q through an angle double the angle of q. In case r is a vector β we have as the product a vector β' which is to be found by rotating conically the vector β about the axis of q through double the angle of q.

It is obvious that operators* of the type $q()q^{-1}$, $r()r^{-1}$, which are called rotators, follow the same laws of multiplication as quaternions, since $q(r()r^{-1})q^{-1} = qr()[qr]^{-1}$. A *gaussian operator* is a rotator multiplied by a numerical multiplier, and is called a *mutation*. The sum of two mutations is not a mutation. As a simple case of rotator we see that if q reduces to a vector α we have as the result of $\alpha\beta\alpha^{-1} = \beta'$ the vector which is the reflection of β in α. The reflection of β in the plane normal to α is evidently

$$-\alpha\beta\alpha^{-1}.$$

EXAMPLES

(1) Successive reflection in two plane mirrors is equivalent

* $q()q^{-1}$ represents a positive orthogonal substitution.

to a rotation about their line of intersection of double their angle.

(2) Successive reflection in a series of mirrors all perpendicular to a common plane, $2h$ in number, making angles in succession (exterior) of φ_{12}, φ_{23}, $\varphi_{34}\cdots$ is equivalent to a rotation about the normal to the given plane to which all are orthogonal, through an angle $\theta = 2h - \pi - 2(\varphi_{12} + \varphi_{34} + \cdots + \varphi_{2h-1,\,2h})$ which is independent of the alternate angles.

(3) Study the case of successive reflections in mirrors in space at any angles.

(4) The types of crystals found in nature and possible under the laws that are found to be true of crystals, are solids such that every face may be produced from a single given face, so far as the angles are concerned, by the following operations:

I, the reversal of a vector, in quaternion form .... -1.
A, rotation about an axis α .... $\alpha^n()\alpha^{-n}$.
IA, rotatory inversion about α .... $-\alpha^n()\alpha^{-n}$.
S, reflection in a plane normal to β ..... $-\beta()\beta^{-1} = \beta()\beta$.

The 32 types of crystals are then generated by the successive combinations of these operations as follows:

Triclinic	C_1	Asymmetric	1.
	C_i	Centrosymmetric	$1, -1$.
Monoclinic	C_s	Equatorial	$1, \beta()\beta$.
	C_2	Digonal polar	$1, \alpha()\alpha^{-1}$.
	C_{2h}	Digonal equatorial	$1, \alpha()\alpha^{-1}, \alpha()\alpha$.
Orthorhombic	C_{2v}	Didigonal polar	$1, \alpha()\alpha^{-1}, \beta()\beta, S\alpha\beta = 0$.
	D_2	Digonal holoaxial	$1, \alpha()\alpha^{-1}, \beta()\beta^{-1}, S\alpha\beta = 0$.
	D_{2h}	Didigonal equatorial	$1, \alpha()\alpha^{-1}, \beta()\beta^{-1}, \alpha()\alpha, S\alpha\beta = 0, A = \alpha^{1/2}()\alpha^{-1/2}$.
Tetragonal	C_4	Tetragonal alternating	$1, -A$.
	D_{2d}	Ditetragonal alternating	$1, -A, \beta()\beta^{-1}$.
	C_4	Tetragonal polar	$1, A$.

	C_{4h}	Tetragonal equatorial	$1,\ A,\ \alpha()\alpha$.
	C_{4v}	Ditetragonal polar	$1,\ A,\ \beta()\beta$.
	D_4	Tetragonal holoaxial	$1,\ A,\ \beta()\beta^{-1}$.
	D_{4h}	Dietragonal equatorial	$1,\ A,\ \alpha()\alpha,\ \beta()\beta^{-1}$.
Rhombohedral	C_3	Trigonal polar	$1,\ B$, where B is $\alpha^{2/3}()\alpha^{-2/3}$.
	C_{3i}	Hexagonal alternating	$1,\ B,\ -B$.
	C_{3v}	Ditrigonal polar	$1,\ B,\ \beta()\beta$.
	D_3	Trigonal holoaxial	$1,\ B,\ \beta()\beta^{-1}$.
	D_{3d}	Dihexagonal alternating	$1,\ B,\ \beta()\beta^{-1},\ \gamma()\gamma,\ \gamma$ bisects $\angle\beta,\ B\beta$.
Hexagonal	C_{3h}	Trigonal equatorial	$1,\ B,\ \alpha()\alpha$.
	D_{3h}	Ditrigonal equatorial	$1,\ B,\ \alpha()\alpha,\ \beta()\beta^{-1}$.
	C_4	Hexagonal polar	$1,\ C$, where $C = \alpha^{1/3}()\alpha^{-1/3}$.
	C_{4h}	Hexagonal equatorial	$1,\ C,\ \alpha()\alpha$.
	C_{4v}	Dihexagonal polar	$1,\ C,\ \beta()\beta$, where $S\alpha\beta = 0$, β bisects angle of γ and $C\gamma$, $S\alpha\gamma = 0$.
	D_4	Hexagonal holoaxial	$1,\ C,\ \beta()\beta^{-1}$.
	D_{6h}	Dihexagonal equatorial	$1,\ C,\ \alpha()\alpha,\ \beta()\beta^{-1}$.
Regular	T	Tesseral polar	$1,\ \alpha()\alpha^{-1},\ \beta()\beta^{-1},\ S\alpha\beta = S\beta\gamma = S\gamma\alpha = 0,\ L$ where $L = (\alpha + \beta + \gamma)()(\alpha + \beta + \gamma)^{-1}$.
	T_h	Tesseral central	$1,\ \alpha()\alpha^{-1},\ \beta()\beta^{-1},\ \gamma()\gamma^{-1},\ L,\ \alpha()\alpha$.
	T_d	Ditesseral polar	$1,\ \alpha()\alpha^{-1},\ \beta()\beta^{-1},\ \gamma()\gamma^{-1},\ L,\ (\alpha + \beta)()(\alpha + \beta)$.
	O	Tesseral holoaxial	$1,\ \alpha()\alpha^{-1},\ \beta()\beta^{-1},\ \gamma()\gamma^{-1},\ L,\ (\alpha + \beta)()(\alpha + \beta)^{-1}$.
	O_h	Ditesseral central	$1,\ \alpha()\alpha^{-1},\ \beta()\beta^{-1},\ \gamma()\gamma^{-1},\ L,\ (\alpha + \beta)()(\alpha + \beta)^{1},\ \alpha()\alpha$.

The student should work out in each case the full set of operators and locate vectors to equivalent points in the various faces.

Ref.—Hilton, *Mathematical Crystallography*, Chap. IV–VIII.

(5) *Spherical Astronomy.* We have the following notation:

λ is a unit vector along the polar axis of the earth,

h is the hour-angle of the meridian,

$L = \cos h/2 + \lambda \sin h/2$,
$i =$ unit vector to zenith,
$j =$ unit vector to south,
$k =$ unit vector to east, $\lambda = i \sin l - j \cos l$, where l is latitude,
$\mu =$ unit vector to intersection of equator and meridian, $\mu = i \cos l + j \sin l$, $S\lambda\mu = Sk\lambda = Sk\mu = 0$,
$d =$ declination of star,
$\delta =$ unit vector to star on the meridian $= \lambda \sin d + \mu \cos d$,
$z =$ azimuth,
$A =$ altitude.
At the hour-angle h, δ becomes $\delta' = L^{-1}\delta L$.

The vertical plane through δ' cuts the horizon in

$$iVi\delta' = jSj\delta' + kSk\delta', \qquad \tan z = Sk\delta'/Sj\delta'.$$

At rising or setting z is found from the condition $Si\delta' = 0$. The prime vertical circle is through i and k. The 6-hour circle is through λ and $V\lambda\mu$.

$a =$ right ascension angle,
$t =$ sidereal time in degrees,
$h = t + a$,
$L_t = \cos t/2 + \lambda \sin t/2$,
$L_a = \cos a/2 + \lambda \sin a/2$,
$\epsilon =$ pole of ecliptic,
$\lambda =$ first point of aries $=$ vernal equinox $= L_t^{-1}\mu L_t$,
$s =$ longitude,
$b =$ latitude,
$M = \cos s/2 + \epsilon \sin s/2$.

Problems. Given l, d, find A and z on 6-hour circle. $S\mu\delta' = 0$.

l, d, find h and z on horizon.

l, d, find A.

l, d, A, find h and z, $\delta' = L^{-1}\delta L = i \cos A + j \cos z + k \sin z$.

l, d, h, find A and z.
a and d, find s and b.

(6) The laws of refraction of light from a medium of index n into a medium of index n' are given by the equation

$$nV\nu\alpha = n'V\nu\alpha'$$

where ν, α, α' are unit vectors along the normal, the incident, and the refracted ray.

The student should show that

$$\alpha' = \nu\sqrt{\left(1 + \frac{n^2}{n'^2}V^2\nu\alpha\right)} - \frac{n}{n'}\nu V\nu\alpha.$$

Investigate two successive refractions, particularly back into the first medium.

(7) It is easy to show that if q and r are any two quaternions, and $\beta = V\cdot VqVr$, we may write

$$q = \frac{V\cdot q\beta}{\beta}, \qquad r = \frac{V\cdot r\beta}{\beta}.$$

(8) For any two quaternions

$$q(q^{-1} \pm r^{-1}) = (r \pm q)r^{-1}, \qquad \text{and} \qquad \frac{1}{\dfrac{1}{q} \pm \dfrac{1}{r}} = r(r \pm q)^{-1}q.$$

(9) If a, b, c are given quaternions we can find a quaternion q that will give three vectors when multiplied by a, b, c resp. That is, we can find q, α, β, γ such that

$$aq = \alpha, \qquad bq = \beta, \qquad cq = \gamma. \quad \text{(R. Russell.)}$$

We have $\alpha = -V\cdot Vc/aVa/b$, etc., or multiples of these.

(10) In a letter of Tait to Cayley, he gives the following:

$$(q + r)()(q + r)^{-1} = (q/r)^x r()r^{-1}(q/r)^{-x}$$
$$= q(q^{-1}r)^y()(q^{-1}r)^{-y}q^{-1} = q^l r^m q^l()q^{-l}r^{-m}q^{-l},$$
$$(Vq + Vr)()(Vq + Vr)^{-1} = (q/r)^{1/2}r()r^{-1}(q/r)^{-1/2},$$

where $\tan xA = a \sin A/(a \cos A + 1)$, $c \sin 2l\alpha \sin m\beta + \cos 2l\alpha \cos m\beta = 2(a \cos \alpha + b \cos \beta)\sqrt{(b \sin \beta)}$, $2c + \sin 2l\alpha \cos m\beta = 2a \sin \alpha/(b \sin \beta)$.

Interpret these formulae.

10. **Products of Several Quaternions.** We will develop some useful formulae from the preceding.

If we multiply $\alpha\beta\cdot\beta\alpha$ we have

$$\alpha^2\beta^2 = S^2\alpha\beta - V^2\alpha\beta.$$

Since $S\alpha x = 0$, if x is a scalar,

$$S\alpha\beta\gamma = S\alpha V\beta\gamma, \qquad S\alpha\beta\gamma\delta = S\alpha V\beta\gamma\delta, \qquad \text{etc.}$$

Since

$$2V\alpha\beta = \alpha\beta - \beta\alpha, \qquad 2S\alpha\beta = \alpha\beta + \beta\alpha,$$

$$\begin{aligned}4V\alpha V\beta\gamma = \alpha\beta\gamma - \alpha\gamma\beta - \beta\gamma\alpha + \gamma\beta\alpha &= 2(\gamma\beta\alpha - \alpha\gamma\beta)\\ &= 2(\gamma\beta\alpha + \gamma\alpha\beta - \alpha\gamma\beta - \gamma\alpha\beta).\end{aligned}$$

For

$$2S\beta\gamma\cdot\alpha = \beta\gamma\alpha + \gamma\beta\alpha = 2\alpha S\beta\gamma = \alpha\beta\gamma + \alpha\gamma\beta,$$

whence

$$\alpha\beta\gamma - \beta\gamma\alpha = \gamma\beta\alpha - \alpha\gamma\beta.$$

Therefore

$$V\alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\alpha\gamma.$$

Adding to each side $\alpha S\beta\gamma$, we have

$$V\alpha\beta\gamma = \alpha S\beta\gamma - \beta S\gamma\alpha + \gamma S\alpha\beta.$$

Since

$$\beta = \alpha^{-1}\alpha\beta, \qquad \beta = \alpha^{-1}S\alpha\beta + \alpha^{-1}V\alpha\beta,$$

which resolves β along and perpendicular to α,

$$Sqrq^{-1} = Sr = qSrq^{-1},$$

$$\begin{aligned}Vqrq^{-1} &= \tfrac{1}{2}(qrq^{-1} - Kq^{-1}KrKq)\\ &= \tfrac{1}{2}(qrq^{-1} - qKrq^{-1}) = qVr\cdot q^{-1}.\end{aligned}$$

That is, if we rotate the field, Sr and TVr are invariant.

Hence $V\alpha\beta\gamma = V\alpha\beta\gamma\alpha\alpha^{-1} = \alpha V\beta\gamma\alpha\cdot\alpha^{-1}$ and $V\alpha\beta\gamma$, $V\beta\gamma\alpha$ are in a plane with α and make equal angles with α. For instance if α, β, γ, $V\alpha\beta\gamma$, $V\beta\gamma\alpha$, $V\gamma\alpha\beta$ intersect a sphere, then α, β, γ bisect the sides of the triangle $V\alpha\beta\gamma$, $V\beta\gamma\alpha$, $V\gamma\alpha\beta$, α being opposite to $V\gamma\alpha\beta$, etc. Evidently if $\alpha_1, \alpha_2\cdots\alpha_n$ are n radii of a sphere forming a polygon, then they bisect the sides of the polygon, given by $V\alpha_1\alpha_2\cdots\alpha_n$, $V\alpha_2\alpha_3\cdots\alpha_n$, $V\alpha_3\cdots\alpha_n\alpha_1\alpha_2$, $\cdots V\alpha_n\alpha_1\cdots\alpha_{n-1}$. This explains the geometrical significance of these vectors. In fact for any vector α and quaternion q, the vector α bisects the angle between $Vq\alpha$ and $V\alpha q$, that is to say we construct $Vq\alpha$ from the vector $V\alpha q$ by reflecting it in α. The same is true for any product, thus $\beta\gamma\delta\epsilon\cdots\nu\alpha$ is different from $\alpha\beta\gamma\delta\epsilon\cdots\nu$ only in the fact that its axis is the reflection in α of the axis of the latter.

$q_2q_3\cdots q_nq_1$ differs from $q_1q_2\cdots q_n$ only in the fact that its axis has been rotated negatively about the axis of q_1 through double the angle of q_1. Indeed

$$q_2q_3\cdots q_nq_1 = q^{-1}(q_1q_2\cdots q_n)q_1.$$

If we apply the formula for expanding $V\alpha V\beta\gamma$ to $V(V\alpha\beta)V\gamma\delta = -V(V\gamma\delta)V\alpha\beta$ we arrive at a most important identity:

$$\begin{aligned} V\cdot V\alpha\beta V\gamma\delta &= \delta S\alpha\beta\gamma - \gamma S\alpha\beta\delta \\ &= -V\cdot V\gamma\delta V\alpha\beta = \alpha S\beta\gamma\delta - \beta S\alpha\gamma\delta. \end{aligned}$$

From this equality we see that for any four vectors

$$\delta S\alpha\beta\gamma = \alpha S\beta\gamma\delta + \beta S\gamma\alpha\delta + \gamma S\alpha\beta\delta.$$

This formula enables us to expand any vector in terms of any three non-coplanar vectors. Again

$$\begin{aligned} \delta S\alpha\beta\gamma - V\beta\gamma S\alpha\delta &= V\cdot\alpha V(V\beta\gamma)\delta \\ &= -V\cdot\alpha V\delta V\beta\gamma = V\alpha\beta S\gamma\delta - V\alpha\gamma S\beta\delta. \end{aligned}$$

We have thus another important formula

$$\delta S\alpha\beta\gamma = V\alpha\beta S\gamma\delta + V\beta\gamma S\alpha\delta + V\gamma\alpha S\beta\delta,$$

enabling us to expand any vector in terms of the three normals to the three planes determined by a set of three vectors, that is, in terms of its normal projections. Since

$$\alpha S\beta\gamma\delta = V\beta\gamma S\alpha\delta + V\gamma\delta S\alpha\beta + V\delta\beta S\alpha\gamma$$

and

$$\beta S\gamma\delta\alpha = V\alpha\gamma S\beta\delta + V\gamma\delta S\alpha\beta + V\delta\alpha S\beta\gamma,$$

we have

$$VV\alpha\beta V\gamma\delta = V\alpha\delta S\beta\gamma + V\beta\gamma S\alpha\delta - V\alpha\gamma S\beta\delta - V\beta\delta S\alpha\gamma.$$

From this we have at once an expansion for $V\alpha\beta\gamma\delta$, namely

$$V\alpha\beta\gamma\delta = V\alpha\beta S\gamma\delta - V\alpha\gamma S\beta\delta + V\alpha\delta S\beta\gamma$$
$$+ S\alpha\beta V\gamma\delta - S\alpha\gamma V\beta\delta + S\alpha\delta V\beta\gamma.$$

Also easily

$$S\alpha\beta\gamma\delta = S\alpha\beta S\gamma\delta - S\alpha\gamma S\beta\delta + S\alpha\delta S\beta\gamma.$$
$$SV\alpha\beta V\gamma\delta = S\alpha\delta S\beta\gamma - S\alpha\gamma S\beta\delta.$$
$$V\cdot\alpha\beta\cdot S\gamma\delta\epsilon = \gamma S\cdot V\alpha\beta V\delta\epsilon - \delta S\cdot V\alpha\beta V\gamma\epsilon + \epsilon S\cdot V\alpha\beta V\gamma\delta$$
$$= -\begin{vmatrix} \gamma & \delta & \epsilon \\ S\alpha\gamma & S\alpha\delta & S\alpha\epsilon \\ S\beta\gamma & S\beta\delta & S\beta\epsilon \end{vmatrix}.$$

In the figure the various points lie on a sphere of radius l. The vectors from the center will be designated by the corresponding Greek letters. The points X, Y, Z are the midpoints of the sides of the $\triangle ABC$. From the figure it is evident that

$$\xi/\beta = \gamma/\xi = (\gamma/\beta)^{1/2}, \qquad \eta/\gamma = \alpha/\eta = (\alpha/\gamma)^{1/2},$$
$$\zeta/\alpha = \beta/\zeta = (\beta/\alpha)^{1/2}.$$

Whence

$$\gamma = \xi\beta\xi^{-1}, \qquad \alpha = \eta\gamma\eta^{-1}, \qquad \beta = \zeta\alpha\zeta^{-1},$$
$$\alpha = \eta\xi\zeta\alpha\zeta^{-1}\xi^{-1}\eta^{-1} = \eta\xi^{-1}\zeta\alpha\zeta^{-1}\xi\eta^{-1} = p\alpha p^{-1},$$

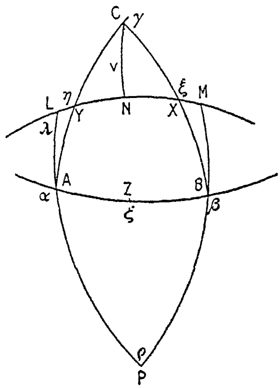

FIG. 18.

where

$$p = \eta\xi^{-1}\zeta,$$

and the axis of p is $\pm\,\alpha$. Also $p\zeta p^{-1} = \eta\xi^{-1}\zeta\xi\eta^{-1}$, so that if P is the pole of the great circle through XY then the rotation $p()p^{-1}$ brings ζ to the same position as the rotation around OP through twice the angle of $\eta\xi^{-1}$. Since ξ goes into ξ' by a rotation about OA as well as one about OP, this means that the new position OZ' is the reflection of OZ in the plane of OPA. The angle of p is then ZAL or ZAP according as the axis is $+\,\alpha$ or $-\,\alpha$. The angles of L and M are right angles, and if we draw CN perpendicular to XY then

$$\triangle NCY = \triangle LAY, \qquad \triangle NCX = \triangle MBX,$$

and

$$AL = BM = CN \quad \text{and} \quad APB \quad \text{is isosceles.}$$

Hence the equal exterior angles at A and B are $ZAL = ZBM = \frac{1}{2}(A + B + C)$.

Draw PZ, then $\angle ZPA = \angle\eta\xi^{-1}$ for it $= \frac{1}{2}\angle BPA = \frac{1}{2}ML = XY$ since $MX = XN$ and $NY = YL$. The angle between the planes LAP and ZOP is thus the biradial $\eta\xi^{-1}$ and also ζ is the biradial whose angle is that of the

planes OAZ, ZOP, so that ZOA and AOL make an angle equal to $\angle p$, hence

$$\angle p = \tfrac{1}{2}(A + B + C).$$

Further

$$p\alpha^{-1} = \eta/\gamma \cdot \gamma/\xi \cdot \zeta/\alpha = (\alpha/\gamma)^{1/2}(\gamma/\beta)^{1/2}(\beta/\alpha)^{1/2} = p'.$$

The angle of p' is thus $\frac{1}{2}(A + B + C - \pi) = \Sigma/2$ where Σ is the spherical excess of $\triangle ABC$.

Consider the quaternion $p = \eta\xi^{-1}\zeta = -\eta\xi\zeta$. The conjugate of p is $Kp = \zeta\xi\eta$, whose axis is also α and angle $-\frac{1}{2}(A + B + C)$. Thus the quaternion $\zeta\xi\eta = -\sin \Sigma/2 - \alpha \cos \Sigma/2$.

Shifting the notation to a more symmetric form we have for any three vectors

$$\begin{aligned}\alpha_1\alpha_2\alpha_3 &= -\sin \Sigma/2 - UV\alpha_1\alpha_2\alpha_3 \cdot \cos \Sigma/2 \\ &= \cos \tfrac{1}{2}\sigma - k \sin \tfrac{1}{2}\sigma,\end{aligned}$$

where Σ is the spherical excess of the triangle the midpoints of whose sides are A_1, A_2, A_3 and σ is the sum of the angles of the triangle. Hence

$$S\alpha_1\alpha_2\alpha_3 = \cos \tfrac{1}{2}\sigma, \qquad V\alpha_1\alpha_2\alpha_3 = -UV\alpha_1\alpha_2\alpha_3 \sin \tfrac{1}{2}\sigma.$$

It is to be noted that the order as written here is for a positive or left-handed cycle from A_1 to A_2 and A_3. Since Σ is the solid angle of the triangle, $-S\cdot\alpha_1\alpha_2\alpha_3$ is the sine of half the solid angle and $-TV\alpha_1\alpha_2\alpha_3$ is the cosine of half the solid angle, made by α_1, α_2, α_3.

If now we have several points as the middle points of the sides of a spherical polygon, say $\alpha_1\alpha_2\cdots\alpha_n$ and the vertex between α_1 and α_n is taken as an origin for spherical arcs drawn as diagonals to the vertices of the polygon, then for the various successive triangles if we call the midpoints of the successive diagonals

$$\zeta_1, \zeta_2, \cdots \zeta_{n-3}$$

we have, taking the axis to the origin which we will call κ, and which is the common axis of all the quaternions made up by the products of three vectors

$$\alpha_1\alpha_2\zeta_1\cdot\zeta_1\alpha_3\zeta_2\cdot\zeta_2\alpha_4\zeta_3\cdots\zeta_{n-3}\alpha_{n-1}\alpha_n = (-)^{n-3}\alpha_1\alpha_2\alpha_3\cdots\alpha_n.$$

The sum of the angles of the polygon is the sum of the angles of all the triangles into which it is divided, so that if this sum is σ we have for any spherical polygon

$$\alpha_1\alpha_2\cdots\alpha_n = (-)^{n-3}[\cos\sigma/2 - \kappa\sin\sigma/2].$$

We are able to say then that if the midpoints of the sides of a spherical polygon are $\alpha_1, \alpha_2, \cdots\alpha_n$, then

$$S\alpha_1\alpha_2\cdots\alpha_n = \pm\cos\sigma/2,$$

where σ is the sum of the angles; the vertices of the polygon are given by

$$UV\alpha_1\alpha_2\cdots\alpha_n, \qquad UV\alpha_2\alpha_3\cdots\alpha_n\alpha_1, \cdots, \\ UV\alpha_n\cdots\alpha_{n-1},$$

each being the vertex whose sides contain the first and last vectors in the product; and the tensors of these vectors are each equal to $\sin\sigma/2$.

The expression $-S\alpha\beta\gamma$ is called the first *staudtian* of $\alpha\beta\gamma$, the *second staudtian* is

$$-SV\alpha\beta V\beta\gamma V\gamma\alpha/TV\alpha\beta TV\beta\gamma TV\gamma\alpha \\ = S^2\alpha\beta\gamma/TV\alpha\beta TV\beta\gamma TV\gamma\alpha,$$

which is evidently the staudtian of the polar triangle.

$$\frac{S\cdot\alpha_1\cdots\alpha_n}{TV\cdot\alpha_1\cdots\alpha_n} = \tan\tfrac{1}{2}\text{ solid angle.}$$

We will summarize here the significance of the expressions worked out thus far, and in particular the meaning of their vanishing.

$S\alpha\beta$ is the product of $T\alpha T\beta$ by the cosine of the angle between α and $-\beta$. It vanishes only if they are perpendicular.

$V\alpha\beta$ is the vector at right angles to both $\alpha\ \beta$ whose length is $T\alpha T\beta$ multiplied by the sine of their angle. It vanishes only if they are parallel.

$S\alpha\beta\gamma$ is the volume of the parallelepiped of $\alpha\ \beta\ \gamma$, taken negatively. It vanishes only if they are all parallel to one plane.

$V\alpha\beta\gamma$, $V\alpha\beta\gamma\delta$, $\cdots$ these vectors are the edges of the polyhedral giving the circumscribed polygon, and if the expression vanishes, we have by separating the quaternion,

$$V\alpha\beta\gamma\delta\cdots = \alpha S\beta\gamma\delta\cdots + V\alpha V\beta\gamma\delta\cdots = 0.$$

Hence α is the axis of $\beta\gamma\delta\cdots$ and $S\beta\gamma\delta\cdots$ equals zero. By changing the vectors cyclically we have n vectors all of which have a zero tensor, so that each edge is the axis of the quaternion of the other $n-1$ taken cyclically. This quaternion in each case has a vanishing scalar.

$n = 3$, $\alpha\ \beta\ \gamma$ are a trirectangular system.

$n = 4$, $\alpha\ \beta\ \gamma\ \delta$ are coplanar, shown by the four vanishing scalars. The angle $\alpha\beta$ = angle $\gamma\delta$.

$n = 5$, the edge $V\alpha\beta\gamma$ is parallel to $V\delta\epsilon$ and cyclically similar parallelisms hold.

We have in all these cases the sum of the angles of the circumscribing polygon a multiple of 2π and it satisfies the inequality $3(n-2)\pi$ is greater than σ which is greater than $(n-2)\pi$. It is evident that if the polygon circumscribed has 540° the vectors lie in one plane.

$S\alpha\beta\gamma\delta = 0$. If $\epsilon = V\alpha\beta\gamma\delta$, then $V\alpha\beta\gamma\delta\epsilon = 0$, and the preceding case is at hand for the five vectors.

$S\cdot\alpha_1\alpha_2\cdots\alpha_n = 0$, the sum of the angles of the polygon is an odd multiple of π.

EXERCISES

1. $S\cdot V\alpha\beta V\beta\gamma V\gamma\alpha = -(S\alpha\beta\gamma)^2$
$V\cdot V\alpha\beta V\beta\gamma V\gamma\alpha = V\alpha\beta(\gamma^2 S\alpha\beta - S\beta\gamma S\gamma\alpha) + \cdots.$

2. $S(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = 2S\alpha\beta\gamma.$

3. $S\cdot V(\alpha + \beta)(\beta + \gamma)V(\beta + \gamma)(\gamma + \alpha)V(\gamma + \alpha)(\alpha + \beta)$
$$= -4(S\cdot\alpha\beta\gamma)^2.$$

4. $S\cdot V(V\alpha\beta V\beta\gamma)(V\beta\gamma V\gamma\alpha)V(V\gamma\alpha V\alpha\beta) = -(S\cdot\alpha\beta\gamma)^4.$

5. $S\cdot\delta\epsilon\zeta = -16(S\cdot\alpha\beta\gamma)^4,$

where
$$\delta = V(V[\alpha + \beta][\beta + \gamma]V[\beta + \gamma][\gamma + \alpha]),$$
$$\epsilon = V(V[\beta + \gamma][\gamma + \alpha]V[\gamma + \alpha][\alpha + \beta]),$$
$$\zeta = V(V[\gamma + \alpha][\alpha + \beta]V[\alpha + \beta][\beta + \gamma]).$$

6. $S(x\alpha + y\beta + z\gamma)(x'\alpha + y'\beta + z'\gamma)(x''\alpha + y''\beta + z''\gamma)$
$$= \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} S\cdot\alpha\beta\gamma.$$

7. $$S\cdot\alpha\beta\gamma\cdot S\cdot\alpha_1\beta_1\gamma_1 = -\begin{vmatrix} S\alpha\alpha_1 & S\beta\alpha_1 & S\gamma\alpha_1 \\ S\alpha\beta_1 & S\beta\beta_1 & S\gamma\beta_1 \\ S\alpha\gamma_1 & S\beta\gamma_1 & S\gamma\gamma_1 \end{vmatrix}.$$

8. $$\begin{vmatrix} S\alpha\alpha_1 & S\alpha\beta_1 & S\alpha\gamma_1 & S\alpha\delta_1 \\ S\beta\alpha_1 & S\beta\beta_1 & S\beta\gamma_1 & S\beta\delta_1 \\ S\gamma\alpha_1 & S\gamma\beta_1 & S\gamma\gamma_1 & S\gamma\delta_1 \\ S\delta\alpha_1 & S\delta\beta_1 & S\delta\gamma_1 & S\delta\delta_1 \end{vmatrix} = 0$$

for any eight vectors. If the element $S\alpha\alpha_1$ is changed to $S\epsilon\alpha_1$ the value is $-S\cdot\beta\gamma\delta\cdot S\cdot\beta_1\gamma_1\delta_1\cdot S\cdot\alpha_1(\epsilon - \alpha)$.

9. $S\cdot V\alpha\beta\gamma V\beta\gamma\alpha V\gamma\alpha\beta = 4S\alpha\beta S\beta\gamma S\gamma\alpha S\alpha\beta\gamma.$

10. From $S^2\rho/\alpha - V^2\rho/\beta = 1$ we find

$$T(S\rho/\alpha + V\rho/\beta) = 1 = T(\tfrac{1}{2}\alpha^{-1}\rho + \tfrac{1}{2}\rho\alpha^{-1} - \tfrac{1}{2}\beta^{-1}\rho + \tfrac{1}{2}\rho\beta^{-1}) = T(\alpha'\rho + \rho\beta')$$

where
$$\alpha' = \tfrac{1}{2}(\alpha^{-1} - \beta^{-1}), \qquad \beta' = \tfrac{1}{2}(\alpha^{-1} + \beta^{-1}).$$

11. If $T\rho = T\alpha = T\beta = 1$ and $S\cdot\alpha\beta\rho = 0$,
$$S\cdot U(\rho - \alpha)U(\rho - \beta) = \pm\tfrac{1}{2}\sqrt{[2(1 - S\alpha\beta)]}.$$

12. If α, β, γ and $\alpha_1, \beta_1, \gamma_1$ are two sets of trirectangular unit vectors such that if $\alpha = \beta\gamma$, $\alpha_1 = \beta_1\gamma$, then we may find angles called Eulerian angles such that

$$\alpha_2 = \alpha\cos\psi + \beta\sin\psi, \qquad \beta_2 = -\alpha\sin\psi + \beta\cos\psi,$$
$$\gamma_3 = \gamma\cos\theta + \alpha_2\sin\theta, \qquad \alpha_3 = -\gamma\sin\theta + \alpha_2\cos\theta,$$
$$\gamma_1 = \gamma_3, \qquad \alpha_1 = \alpha_3\cos\varphi + \beta_2\sin\varphi,$$
$$\beta_1 = -\alpha_3\sin\varphi + \beta_2\cos\varphi.$$

13. If $q = \alpha_1\alpha_2 \cdots \alpha_n$ then if we reflect an arbitrary vector in succession in $\alpha_n, \alpha_{n-1}, \cdots \alpha_2\alpha_1$ when $Sq = 0$ the final position will be a simple reflection of ρ in a fixed vector, and if $Vq = 0$ the final position will be on the line of ρ itself. Similar statements hold if the reflections are in planes that are normal respectively to $\alpha_n, \cdots \alpha_1$.

11. Functions. We notice some expressions now of the nature of functions of a quaternion. We have the following identity which is useful:

$$(\alpha\beta)^n + (\beta\alpha)^n = (\alpha\beta + \beta\alpha)[(\alpha\beta)^{n-1}] - \alpha\beta\beta\alpha[(\alpha\beta)^{n-2} + (\beta\alpha)^{n-2}]$$
$$= 2S\alpha\beta[(\alpha\beta)^{n-1} + (\beta\alpha)^{n-1}] - \alpha^2\beta^2[(\alpha\beta)^{n-2} + (\beta\alpha)^{n-2}].$$

Whence $2^nS^n\alpha\beta = (\alpha\beta + \beta\alpha)^n = [(\alpha\beta)^n + (\beta\alpha)^n]$

$$+ \frac{n!}{1!(n-1)!}[(\alpha\beta)^{n-2} + (\beta\alpha)^{n-2}]\alpha^2\beta^2$$

$$+ \frac{n!}{2!(n-2)!}[(\alpha\beta)^{n-4} + (\beta\alpha)^{n-4}]\alpha^4\beta^4 + \cdots$$

$$= 2S(\alpha\beta)^n + \frac{n!}{1!(n-1)!}\alpha^2\beta^2S(\alpha\beta)^{n-2}\cdots.$$

This implies the familiar formula for the expansion of $\cos^n \theta$ in terms of $\cos n\theta$, $\cos (n-2)\theta$, and we can write as the reverse formula

$$S(\alpha\beta)^n = (-)^{n/2}[\alpha^n\beta^n - n^2S^2\alpha\beta\cdot\alpha^{n-2}\beta^{n-2}/2! + n^2(n^2-2^2)S^4\alpha\beta\cdot\alpha^{n-4}\beta^{n-4}/4! - \cdots]\ n \text{ even}$$
$$(-)^{(n-1)/2}[nS\alpha\beta\cdot\alpha^{n-1}\beta^{n-1}/1! - n(n^2-1^2)S^3\alpha\beta\cdot\alpha^{n-3}\beta^{n-3}/3! + \cdots]\ n \text{ odd}.$$

Likewise

$$TV^{2n}\alpha\beta = (-1)^n/2^{2n-1}[S(\alpha\beta^{2n} - \frac{(2n)!}{1!(2n-1)!}S(\alpha\beta^{2n-2}\alpha^2\beta^2 + \cdots]$$

$$TV^{2n-1}\alpha\beta = (-1)^n/2^{2n-2}[TV(\alpha\beta)^{2n-1} - \frac{(2n-1)!}{1!(2n-2)!}TV(\alpha\beta)^{2n-3} + \cdots]$$

$$TV(\alpha\beta)^n/TV\alpha\beta = (-)^{n/2}[nS\alpha\beta\cdot\alpha^{n-2}\beta^{n-2}/1! - n(n^2-2^2)S^3\alpha\beta\alpha^{n-4}\beta^{n-4}/3! + \cdots]\ n \text{ even}$$
$$(-1)^{(n-1)/2}[1 - (n^2-1^2)S^2\alpha\beta\cdot\alpha^{n-3}\beta^{n-3}/2! + \cdots]\ n \text{ odd.}$$

Since β/α is a quaternion whose powers have the same axis we have $(1-\beta/\alpha)^{-1} = 1 + \beta/\alpha + (\beta/\alpha)^2 + \cdots$ when $T\beta < T\alpha$, and taking the scalar gives the well-known formula

$$S\frac{\alpha}{\alpha-\beta} = 1 + S\beta/\alpha + S(\beta/\alpha)^2 + \cdots.$$

Likewise

$$TV\frac{\alpha}{\alpha-\beta} = TV\beta/\alpha + TV(\beta/\alpha)^2 + \cdots.$$

If we define the logarithm as in theory of functions of a complex variable we have

$$\begin{aligned}\log(1-\beta/\alpha) &= \log T(1-\beta/\alpha) + \log U(1-\beta/\alpha)\\ &= -\beta/\alpha - \tfrac{1}{2}(\beta/\alpha)^2 - \tfrac{1}{3}(\beta/\alpha)^3 - \cdots.\end{aligned}$$

Therefore

$$\log T(1-\beta/\alpha) = -S\beta/\alpha - \tfrac{1}{2}S(\beta/\alpha)^2 - \cdots$$

$$\angle\frac{\alpha-\beta}{\alpha} = TV\log(1-\beta/\alpha) = TV\beta/\alpha + \tfrac{1}{2}TV(\beta/\alpha)^2\cdots.$$

Again

$$T(\alpha-\beta)^{-1} = T\alpha^{-1} - T(1-\beta/\alpha)^{-1} = T\alpha^{-1}[1 + P_1(-SU\beta/\alpha)T\beta/\alpha + P_2(-SU\beta/\alpha)T^2\beta/\alpha + \cdots],$$

where P_1 P_2 are the Legendrian polynomials.

Evidently for coaxial quaternions we have the whole theory of functions of a complex variable applicable.

12. Solution of Some Simple Equations.

(1). If $\alpha\rho = a$ then $\rho = \alpha^{-1}a$.

(2). If $S\alpha\rho = a$ then we set $V\alpha\rho = \zeta$ where ζ is any vector perpendicular to α, and adding, $\rho = a\alpha^{-1} + \alpha^{-1}\zeta$.

(3). If $V\alpha\rho = \beta$ then $S\alpha\rho = x$ where x is any scalar, and adding we have $\rho = \alpha^{-1}\beta + x\alpha^{-1}$.

(4). If $V\alpha\rho\beta = \gamma$ then $S\alpha V\alpha\rho\beta = S\alpha^2\rho\beta = \alpha^2 S\rho\beta = S\alpha\gamma$ and $S\beta V\alpha\rho\beta = \beta^2 S\alpha\rho = S\beta\gamma$. Now

$$V\alpha\rho\beta = \alpha S\beta\rho - \rho S\alpha\beta + \beta S\alpha\rho$$

and substituting we have

$$\rho = [\alpha^{-1}S\alpha\gamma + \beta^{-1}S\beta\gamma - \gamma]/S\alpha\beta.$$

The solution fails if $S\alpha\beta = 0$. In this case the solution is

$$\rho = -\alpha^{-1}S\beta^{-1}\gamma - \beta^{-1}S\alpha^{-1}\gamma + xV\alpha\beta,$$

x any scalar.

(5). If $V\alpha\beta\rho = \gamma$ then $S\alpha\beta\rho S\alpha\beta = S\alpha\beta\gamma$ and $S\alpha\beta\rho = S\alpha\beta\gamma/S\alpha\beta$. Adding to $V\alpha\beta\rho$, we have

$$\alpha\beta\rho = \gamma + S\alpha\beta\gamma/S\alpha\beta \text{ and } \rho = \beta^{-1}\alpha^{-1}\gamma + \beta^{-1}\alpha^{-1}S\alpha\beta\gamma/S\alpha\beta.$$

(6). If $S\alpha\rho = a$, $S\beta\rho = b$, then $\alpha^2\beta^2\rho = xV\alpha\beta + V(a\beta - b\alpha)V\alpha\beta$.

(7). If $S\alpha\rho = a$, $S\beta\rho = b$, $S\gamma\rho = c$, then

$$\rho S\alpha\beta\gamma = aV\beta\gamma + bV\gamma\alpha + cV\alpha\beta.$$

(8). If $q\alpha q^{-1} = \beta$ then $q = (x\beta + y)/(\alpha + \beta)$ where x and y are any scalars. Or we may write

$$q = u + v(\alpha + \beta) + wV\alpha\beta \quad \text{where} \quad u = -wS\alpha(\alpha + \beta).$$

(9). If $q\alpha q^{-1} = \gamma$, $q\beta q^{-1} = \delta$, then

$$q = x\left[1 + \frac{V(\gamma - \alpha)(\delta - \beta)}{S(\gamma + \alpha)(\delta - \beta)}\right].$$

(10). If $q\alpha q^{-1} = \xi$, $q\beta q^{-1} = \eta$, $q\gamma q^{-1} = \zeta$, then

$$S\cdot q(\xi - \alpha) = 0, \qquad S\cdot q(\eta - \beta) = 0, \qquad S\cdot q(\zeta - \gamma) = 0,$$

hence Vq is coplanar with the parentheses, and we have

$$x(\xi - \alpha) + y(\eta - \beta) + z(\zeta - \gamma) = 0$$

where

$$x : y : z = -2S\gamma(\eta - \beta) : 2S\gamma(\xi - \alpha) : S(\xi + \alpha)(\eta - \beta).$$

The six vectors are not independent. Vq is easily found and thence Sq from

$$q\alpha = \xi q.$$

(11). If $(\rho - \alpha)^{-1} + (\rho - \beta)^{-1} - (\rho - \gamma)^{-1} - (\rho - \delta)^{-1} = 0$, then if we let

$$(\rho' - \alpha')^{-1} = 1 \div ([(\rho - \delta)^{-1} - \delta] - [(\alpha - \delta)^{-1} - \delta])$$
$$= (\rho - \delta)(\rho - \alpha)^{-1}(\alpha - \delta), \text{ etc.},$$

where ρ', α', β', γ' are the vectors from D, the extremity of δ, to the inverses with respect to D, of the extremities of ρ, α, β, γ, then

$$(\rho' - \alpha')^{-1} + (\rho' - \beta')^{-1} - (\rho' - \gamma')^{-1} = 0.$$

Prove that

$$\frac{\rho' - \beta'}{\rho' - \alpha'} = \frac{\gamma' - \beta'}{\rho' - \gamma'} = \frac{\rho' - \gamma'}{\gamma' - \alpha'} = \left[\frac{\gamma' - \beta'}{\gamma' - \alpha'}\right]^{1/2},$$

whence ρ' and ρ. (R. Russell.)

(12). If $(q - a)^{-1} + (q - b)^{-1} - (q - c)^{-1} - (q - d)^{-1} = 0$, we set

$$(q - d)(q' - d) = (a - d)(a' - d) = (b - d)(b' - d)$$
$$= (c - d)(c' - d) = 1,$$

thence

$$(q-d)^{-1}-(q-a)^{-1}=(q-d)^{-1}(a-d)(q-a)^{-1}$$

$$(q-d)^{-1}[(a-d)/(q-a)+(b-d)/(q-b)-(c-d)/(q-c)]$$
$$=(q'-a')^{-1}+(q'-b')^{-1}-(q'-c')^{-1}$$

and we have q' from

$$(b'-c')/(q'-c')=(q'-b')/(q'-a')$$
$$=(q'-c')/(a'-c')=[(b'-c')/(a'-c')]^{\frac{1}{2}}.$$

(R. Russell.)

13. **Characteristic Equation.** If we write $q = Sq + Vq$ and square both sides we have $q^2 = S^2q + (Vq)^2 + 2Sq\cdot Vq$ whence

$$q^2 - 2qSq + S^2q - V^2q = 0.$$

This equation is called the *characteristic equation* of q. The coefficients

$$2Sq \qquad \text{and} \qquad S^2q - V^2q = T^2q$$

are the invariants of q; they are the same, that is to say, if q is subjected to the rotation $r()r^{-1}$. They are also the same if Kq is substituted for q. Hence they will not define q but only any one of a class of quaternions which may be derived from each other by the group of all rotations of the form $r()r^{-1}$ or by taking the conjugate.

The equation has two roots in general,

$$Sq + Tq\sqrt{-1} \qquad \text{and} \qquad Sq - Tq\sqrt{-1}.$$

Since these involve the $\sqrt{-1}$ it leads us to the algebra of biquaternions which we do not enter here, but a few remarks will be necessary to place the subject properly.

Since the invariants do not determine q we observe that we must also have UVq in order to have the other two parameters involved.

If we look upon UVq as known then we may write the roots of the characteristic equation in the number field of quaternions as $Sq + TVqUVq$ and $Sq - TVqUVq$ or

$$q \qquad \text{and} \qquad Kq.$$

If we set $q + r$ for q and expand, afterwards drop all the terms that arise from the identical equations of q and r separately, we have left the *characteristic equation of two quaternions,* which will reduce to the first form when they are made to be equal. This equation is

$$qr + rq - 2Sq \cdot r - 2Sr \cdot Vq + 2SqSr - 2SVqVr = 0.$$

We might indeed start with this equation and develop the whole algebra from it.

We may write it

$$qr + rq - 2qSr - 2rSq + 4Sq \cdot Sr + S \cdot qr + S \cdot rq = 0$$

which involves only the scalars of q, r, qr, and rq.

14. **Biquaternions.** We should notice that if the parameters involved in q can be imaginary or complex then division is no longer unique in certain cases. Thus if

$$Q^2 = q^2$$

we have as possible solutions

$$Q = \pm q \qquad \text{and also } Q = \pm \sqrt{(-1)}\,UVq \cdot q.$$

If $q^2 = 0$ and $Vq = 0$ then $TVq = 0$ and we have $Vq = x(i + j\sqrt{-1})$ where x is any scalar and i, j are any two perpendicular unit vectors.

CHAPTER VII

APPLICATIONS

1. The Scalar of Two Vectors

1. **Notations.** The scalar of the product of two vectors is defined independently by writers on vector algebra, as a *product*. In such cases the definition is usually given for the negative of the scalar since this is generally essentially positive. A table of current notations is given. If α and β define two fields, we shall call $S\cdot\alpha\beta$ the *virial* of the two fields.

$S\cdot\alpha\beta = -\alpha\times\beta$	Grassman, Resal, Somoff, Peano, Burali-Forti, Marcolongo, Timerding.
$-\alpha\cdot\beta$	Gibbs, Wilson, Jaumann, Jung, Fischer.
$-\alpha\beta$	Heaviside, Silberstein, Föppl, Ferraris, Heun, Bucherer.
$-(\alpha\beta)$	Bucherer, Gans, Lorentz, Abraham, Henrici.
$-\alpha\vert\beta$	Grassman, Jahnke, Fehr, Hyde.
Cos $\alpha\beta$	Macfarlane.
$[\alpha\beta]$	Caspary.

For most of these authors, the scalar of two vectors, though called a product, is really a function of the two vectors which satisfies certain formal laws. While it is evident that any one may arbitrarily choose to call any function of one or more vectors their product, it does not seem desirable to do so. For Gibbs, however, the scalar is defined to be a function of the dyad of the two vectors, which dyad is a real product. The dyad or dyadic of Gibbs, as well as the vectors of most writers on vector analysis, are not considered to be numbers or hypernumbers.

They are looked upon as geometric or physical entities, from which by various modes of "combination" or determination other geometric entities are found, called products. The essence of the Hamiltonian point of view, however, is the definition by means of geometric entities of a system of hypernumbers subject to one mode of multiplication, which gives hypernumbers as products. Functions of these products are considered when useful, but are called functions.

2. **Planes and Spheres.** It is evident that the condition for orthogonality will yield several useful equations, and of these we will consider a few.

The plane through a point A, whose vector is α, perpendicular to a line whose direction is δ has for its equation, since $\rho - \alpha$ is any vector in the plane,

$$S\cdot\delta(\rho - \alpha) = 0.$$

If we set $\rho = \delta S\alpha/\delta$ we have the equation satisfied and as this vector is parallel to δ it is the perpendicular from the origin to the plane. The perpendicular from a point B is $\delta^{-1}S(\alpha - \beta)\delta$.

If a sphere has center D and radius $T\beta$ where β and $-\beta$ are the vectors from the center to the extremities of a diameter, then the equation of the sphere is given by the equation

$$S(\rho - \delta + \beta)(\rho - \delta - \beta) = 0, \text{ or } \rho^2 - 2S\delta\rho + \delta^2 - \beta^2 = 0.$$

The plane through the intersection of the two spheres

$$\rho^2 - 2S\delta_1\rho + c_1 = 0 = \rho^2 - 2S\delta_2\rho + c_2$$

is

$$2S(\delta_1 - \delta_2)\rho = c_1 - c_2.$$

The form of this equation shows that it represents a plane

perpendicular to the center line of the spheres. The point where it crosses this line is

$$\rho = \frac{x_1\delta_1 + x_2\delta_2}{x_1 + x_2},$$

whence solving, we find

$$\rho = V(\delta_1 + \delta_2)^{-1}(V\delta_1\delta_2 + \tfrac{1}{2}(c_1 - c_2)).$$

3. **Virial.** If β is the representative of a force in direction and magnitude then its projection on the direction α is $\alpha^{-1}S\alpha\beta$, and perpendicular to this direction $\alpha^{-1}V\alpha\beta$. If α is in the line of action of the force, the projection is β. If α is a direction not in the line of action then the projection gives the component of the force in the direction α. If α is the vector to the point of application of the force then $S\alpha\beta$ is the *virial* of the force with respect to α, a term introduced by Clausius. It is the work that would be done by the force in moving the point of application through the vector distance α. If α is an infinitesimal distance say, $\delta\alpha$, then $-S\delta\alpha\beta$ is the virtual work of a small virtual displacement. The total virtual work would be $\delta V = -\Sigma S\delta\alpha_n\beta_n$ for all the forces.

4. **Circulation.** In case a particle is in a vector field (of force, or velocity, or otherwise) and it is subjected to successive displacements $\delta\rho$ along an assigned path from A to B, we may form the negative scalar of the vector intensity of the field and the displacement. If the vector intensity varies from point to point the displacements must be infinitesimal. The sum of these products, if there is a finite number, or the definite integral which is the limit of the sum in the infinitesimal case, is of great importance. If a point is moving with a velocity σ [cm./sec.] in a field of force of β dynes, the *activity* of the field on the point is

$-S \cdot \beta\sigma$ [ergs/sec.]. The field may move and the point remain stationary, in which case the activity is $S \cdot \beta\sigma$. The activity is also called the *effect,* and the *power*. If σ is the vector function of ρ which gives the field at the point P we have for the sum

$$-\Sigma S\sigma\delta\rho \qquad \text{or} \qquad -\int_A^B S\sigma\delta\rho.$$

This integral or sum is called the *circulation of the path for the field* σ.

5. **Volts, Gilberts.** For a force field the circulation is the work done in passing from A to B. If the field is an electric field $\mathbf{E}$, the circulation is the difference in voltage between A and B. If the field is a magnetic field $\mathbf{H}$, then the circulation is the difference in gilbertage from A to B. It is measured in gilberts, the unit of magnetic field being a gilbert per centimeter. There is no name yet approved for the unit of the electrostatic field, and we must call it volt per centimeter. The unit of force is the dyne and of work the erg.

6. **Gausses and Lines.** In case the field is a field of flux σ, and the vector $U\nu$ is the outward normal of a surface through which the flux passes, then

$$-S\sigma U\nu$$

is the intensity of flux normal to or through the surface per square centimeter. The unit of magnetostatic flux $\mathbf{B}$ is called a *gauss;* the unit of electrostatic flux $\mathbf{D}$ is called a *line*. The total flux through a finite surface is the areal integral

$$-\int S\sigma U\nu dA, \qquad \text{written also} \qquad -\int S\sigma d\nu.$$

The flux-integral is called the transport or the discharge. Thus if $\mathbf{D}$ is the electric induction or displacement, the

discharge through a surface A is $-\int S\mathbf{D}U\nu dA$, measured in coulombs. Similarly for the magnetic induction $\mathbf{B}$, the discharge is measured in maxwells.

7. **Energy-Density. Activity-Density.** Among other scalar products of importance we find the following. If $\mathbf{E}$ and $\mathbf{D}$ are the electric intensity in volts/cm. and induction in lines at a point, $-\frac{1}{2}S\mathbf{ED}$ is the energy-density in the field at the point in joules/cc. If $\mathbf{H}$ and $\mathbf{B}$, likewise, are the magnetic intensity in gilberts/cm., and gausses, respectively, $-\frac{1}{2\pi}S\mathbf{HB}$ is the energy in ergs. If $\mathbf{J}$ is the electric current-density in amperes/cm.^{2}, $-S\cdot\mathbf{EJ}$ is the activity in watts/cc. If $\mathbf{G}$ is the magnetic current-density in heavisides*/cm.^{2}, $-S\cdot\mathbf{HG}$ is the activity in ergs/sec. If the field varies also, the electric activity is $-S\cdot\mathbf{E}(\mathbf{J}+\dot{\mathbf{D}})$ and the magnetic activity $-S\cdot\mathbf{H}(\mathbf{G}+\dot{\mathbf{B}})$.

EXERCISES

1. An insect has to crawl up the inside of a hemispherical bowl, the coefficient of friction being 1/3, how high can it get?

2. The force of gravity may be expressed in the form $\sigma = -mgk$. Show that the circulation from A to B is the product of the weight by the vertical difference of level of A and B.

3. If the force of attraction of the earth is $\sigma = -hU\rho/\rho^2$ show that the work done in going from A to B is

$$h[T\alpha^{-1} - T\beta^{-1}].$$

4. The magnetic field at a distance a from the central axis of an infinite straight wire carrying a current of electricity of I amperes is

$\mathbf{H} = 0.2Ia^{-1}(-\sin\theta i + \cos\theta j)$ (i and j perpendicular to wire)

and the differential tangent to a circle of radius a is $(-a\sin\theta\, i + a\cos\theta\, j)d\theta$. Show that the gilbertage is $0.2I(\theta_2 - \theta_1)$ gilberts, which for one turn is $0.4\pi I$.

Prove that we get the same result for a square path.

5. The permittivity κ of a specimen of petroleum is 2 [abfarad/cm.], and on a small sphere is a charge of 0.0001 coulomb. The value of the displacement $\mathbf{D}$ at the point ρ is then

$$\mathbf{D} = \frac{0.002}{4\pi}U\rho/T\rho^2 \qquad \text{[lines]}.$$

* A heaviside is a magnetic current of 1 maxwell per second.

What is the discharge through an equilateral triangle whose corners are each 4 cm. from the origin, the plane of the triangle perpendicular to the field?

6. If magnetic inductivity μ is 1760 [henry/cm.] and a magnetic field is given by

$$\mathbf{H} = 7\alpha \qquad \text{[gilbert/cm.]},$$

then the magnetic induction is

$$\mathbf{B} = 7\cdot 1760\alpha \qquad \text{[gausses]}.$$

What is the flux through a circular loop of radius a crossing the field at an angle of 30°?

7. If the velocity of a stream is given by

$$\sigma = 24(\cos\theta\, i + \sin\theta\, j),$$

what is the discharge per second through a portion of the plane whose equation is $Si\rho = -12$ from

$$\theta = 10° \qquad \text{to} \qquad \theta = 20°?$$

8. The electric induction due to a charge at the origin of e coulombs is

$$\mathbf{D} = -eU\rho/T\rho^2 4\pi \qquad \text{[lines]}.$$

What is the total flux of induction through a parallelepiped whose center is the origin?

9. The magnetic induction due to a magnetic point of m maxwells is

$$\mathbf{B} = -mU\rho/T\rho^2 \qquad \text{[gausses]}.$$

What is the total flux of induction through a sphere whose center is the point?

10. In problem 8, if the permittivity is $2 = \kappa$, then the electric intensity

$$\mathbf{E} = \kappa^{-1}\mathbf{D}\cdot 4\pi.$$

What is the amount of energy enclosed in a sphere of radius 3 cm. and center at a distance from the origin of 10 cm.?

11. In problem 9, if the inductivity is 1760 and the magnetic intensity is

$$\mathbf{H} = \mu^{-1}\mathbf{B},$$

how much energy is enclosed in a box 2 cm. each way, whose center is 10 cm. from the point and one face perpendicular to the line joining the point and the center?

12. If the current in a wire 1 mm. in diameter is 10 amperes and the drop in voltage is 0.001 per cm., what is the activity?

13. If there is a leakage of 10 heavisides through a magnetic area of 4 cm.², and the magnetic field is 5 gilberts/cm., what is the activity?

14. Through a circular spot in the bottom of a tank which is kept level full of water there is a leakage of 100 cc. per second, the spot having an area of 20 cm.². If the only force acting is gravity what is the activity?

15. If an electric wave front from the sun has in its plane surface an electric intensity of 10 volts per cm., and a magnetic intensity of 0.033 gilberts per cm., and if for the free ether or for air $\mu = 1$ and $\kappa = \frac{1}{9}\cdot 10^{-20}$, what is the energy per cc. at the wave front? (The average energy is half this maximum energy and is according to Langley $4.3\cdot 10^{-5}$ ergs per cc. per sec.)

16. If a charge of e coulombs is at a point A and a magnetic point at B has m maxwells, what is the energy per cc. at P, any point in space, the medium being air?

8. Geometric Loci in Scalar Equations.

(1). The equation of the sphere may be written in each of the forms

$$
\begin{aligned}
\alpha/\rho &= K\rho/\alpha,\\
S(\rho-\alpha)/(\rho+\alpha) &= 0,\\
S2\alpha/(\rho+\alpha) &= 1,\\
S2\rho/(\rho+\alpha) &= 1,\\
T(S\rho/\alpha + V\rho/\alpha) &= 1,\\
T(\rho - c\alpha) &= T(c\rho-\alpha),\\
S(\rho-\alpha)(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\rho) &= 0,\\
\alpha^2 S\beta\gamma\rho + \beta^2 S\gamma\alpha\rho + \gamma^2 S\alpha\beta\rho &= \rho^2 S\alpha\beta\gamma,
\end{aligned}
$$

$$
\begin{vmatrix}
0 & (\rho-\alpha)^2 & (\rho-\beta)^2 & (\rho-\gamma)^2 & (\rho-\delta)^2\\
(\rho-\alpha)^2 & 0 & (\alpha-\beta)^2 & (\alpha-\gamma)^2 & (\alpha-\delta)^2\\
(\rho-\beta)^2 & (\beta-\alpha)^2 & 0 & (\beta-\gamma)^2 & (\beta-\delta)^2\\
(\rho-\gamma)^2 & (\gamma-\alpha)^2 & (\gamma-\beta)^2 & 0 & (\gamma-\delta)^2\\
(\rho-\delta)^2 & (\delta-\alpha)^2 & (\delta-\beta)^2 & (\delta-\gamma)^2 & 0
\end{vmatrix} = 0.
$$

Interpret each form.

(2). The equation of the ellipsoid may be written in the forms

$$S^2\rho/\alpha - V^2\rho/\beta = 1,$$

where α is not parallel to β,

$$
\begin{aligned}
T(\rho/\gamma + K\rho/\delta) &= T(\rho/\delta + K\rho/\gamma),\\
T(\mu\rho + \rho\lambda) &= \lambda^2 - \mu^2.
\end{aligned}
$$

The planes

$$S\frac{\rho}{\alpha} = \pm S\frac{\rho}{\beta}$$

cut the ellipsoid in circular sections on $T\rho = T\beta$. These are the cyclic planes. $T\beta$ is the mean semi-axis, $U\beta$ the axis of the cylinder of revolution circumscribing the ellipsoid. α is normal to the plane of the ellipse of contact of the cylinder and the ellipsoid.

In the second form let

$$\delta^{-1} = -\frac{\lambda}{t^2}, \qquad \gamma^{-1} = -\frac{\mu}{t^2}, \qquad t^2 = T\lambda^2 - T\mu^2,$$

then the semi-axes are

$$a = T\lambda + T\mu, \qquad b = \frac{T\lambda^2 - T\mu^2}{T(\lambda - \mu)}, \qquad c = T\lambda - T\mu.$$

(3). The hyperboloid of two sheets is $S^2\rho/\alpha + V^2\rho/\beta = 1$.
(4). The hyperboloid of one sheet is $S^2\rho/\alpha + V^2\rho/\beta = -1$.
(5). The elliptic paraboloid of revolution is

$$S\rho/\beta + V^2\rho/\beta = 0.$$

(6). The elliptic paraboloid is $S\rho/\alpha + V^2\rho/\beta = 0$.
(7). The hyperbolic paraboloid is $S\rho/\alpha\, S\rho/\beta = S\rho/\gamma$.
(8). The torus is

$$T(\pm\, bU\alpha^{-1}V\alpha\rho - \rho) = a,$$
$$2bTV\alpha\rho = \pm\,(T\rho^2 + b^2 - a^2),$$
$$4b^2S^2\alpha\rho = 4b^2T^2\rho - (T^2\rho + b^2 - a^2)^2,$$
$$4a^2T^2\rho - 4b^2S^2\alpha\rho = (T^2\rho - b^2 + a^2)^2,$$
$$SU(\rho - \alpha\sqrt{(a^2 - b^2)})/(\rho + \alpha\sqrt{(a^2 - b^2)}) = \pm\, b/a,$$
$$\rho = \pm\, bU\alpha^{-1}V\alpha\tau + aU\tau, \qquad \tau \text{ any vector.}$$

(9). Any surface is given by

$$\rho = \varphi(u, v).$$

A developable is given by $\rho = \varphi(t) + u\varphi'(t)$.

(10). A cone is $f(U[\rho - \alpha]) = 0$.

The quadric cone is $S\alpha\rho S\beta\rho - \rho^2 = 0$.

The cone through α, β, γ, δ, ϵ is

$$S \cdot V(V\alpha\beta V\delta\epsilon)V(V\beta\gamma V\epsilon\rho)V(V\gamma\delta V\rho\alpha) = 0,$$

which is Pascal's theorem on conics.

The cones of revolution through λ, μ, ν are

$$\frac{1}{T\rho} S\rho\left(\Sigma \pm \frac{V\mu\nu}{SV\mu\nu U\lambda}\right) = 1.$$

The cones of revolution which touch $S\lambda\rho = 0$, $S\mu\rho = 0$, $S\nu\rho = 0$, are

$$TV\rho^{-1}V\rho\left(\Sigma \pm \frac{V\mu\nu}{SU\lambda V\mu\nu}\right) = 1.$$

The cone tangent to $(\rho - \alpha)^2 + c^2 = 0$ from β is

$$c^2(\rho - \alpha - \beta)^2 = V^2\beta(\rho - \alpha).$$

The polar plane of β is $S\beta(\rho - \alpha) = -c^2$.

The cone tangent to

$$S^2\frac{\rho}{\alpha} - V^2\frac{\rho}{\beta} = 1$$

from γ is

$$\left(S^2\frac{\rho}{\alpha} - V^2\frac{\rho}{\beta} - 1\right)\left(S^2\frac{\gamma}{\alpha} - V^2\frac{\gamma}{\beta} - 1\right) - \left(S\frac{\rho}{\alpha}S\frac{\gamma}{\alpha} - SV\frac{\rho}{\alpha}V\frac{\gamma}{\alpha} - 1\right)^2 = 0.$$

The cylinder with elements parallel to γ is

$$\left(S^2\frac{\rho}{\alpha} - V^2\frac{\rho}{\beta} - 1\right)\left(S^2\frac{\gamma}{\alpha} - V^2\frac{\gamma}{\beta}\right) - \left(S\frac{\rho}{\alpha}S\frac{\gamma}{\alpha} - SV\frac{\rho}{\alpha}V\frac{\gamma}{\alpha}\right)^2 = 0.$$

For further examples consult Joly: Manual of Quaternions.

2. The Vector of Two Vectors

Notations. If α and β are two fields, we shall call $V \cdot \alpha\beta$ the *torque* of the two fields.

$V\alpha\beta = V\alpha\beta$	Hamilton, Tait, Joly, Heaviside, Föppl, Ferraris, Carvallo.
$\alpha\beta$	Grassman, Jahnke, Fehr.
$\alpha \times \beta$	Gibbs, Wilson, Fischer, Jaumann, Jung.
$[\alpha, \beta]$	Lorentz, Gans, Bucherer, Abraham, Timerding.
$[\alpha \| \beta]$	Caspary.
$\alpha \wedge \beta$	Burali-Forti, Marcolongo, Jung.
$\overline{\alpha\beta}$	Heun.
$\text{Sin}\,\alpha\beta$	Macfarlane.
$I_A\alpha_B$	Peano.

1. **Lines.** The condition that two lines be parallel is that $V\alpha\beta = 0$. Therefore the equation of the line through the origin in the direction α is $V\alpha\rho = 0$.

The line through β parallel to α is $V\alpha(\rho - \beta) = 0$ or $V\alpha\rho = V\alpha\beta = \gamma$. The perpendicular from δ on the line $V\alpha\rho = \gamma$ is

$$- \alpha^{-1}V\alpha\delta + \alpha^{-1}\gamma.$$

The line of intersection of the planes, $S\lambda\rho = a$, $S\mu\rho = b$, is $V\rho V\lambda\mu = a\mu - b\lambda$. If we have lines $V\rho\alpha = \gamma$ and $V\rho\beta = \delta$ then a vector from a point on the first to a point on the second is $\delta\beta^{-1} - \gamma\alpha^{-1} + x\beta - y\alpha$. If now the lines intersect then we can choose x and y so that this vector will vanish, corresponding to the two coincident points, and thus

$$S(\delta\beta^{-1} - \gamma\alpha^{-1})\beta\alpha = 0 = S\delta\alpha + S\gamma\beta.$$

If we resolve the vector joining the two points parallel and perpendicular to $V\alpha\beta$ we have*

$$
\begin{aligned}
\delta\beta^{-1} - \gamma\alpha^{-1} + x\beta - y\alpha
&= (V\alpha\beta)^{-1}S\cdot V\alpha\beta(\delta\beta^{-1} - \gamma\alpha^{-1} + x\beta - y\alpha) \\
&\quad + (V\alpha\beta)^{-1}V\cdot V\alpha\beta(\delta\beta^{-1} - \gamma\alpha^{-1} + x\beta - y\alpha) \\
&= -(V\alpha\beta)^{-1}(S\delta\alpha + S\beta\gamma) \\
&\quad - \alpha^{-1}\left[S\frac{\gamma S\alpha\beta - \delta\alpha^2}{V\alpha\beta} + y\alpha^2\right] \\
&\quad - \beta^{-1}\left[S\frac{\delta S\alpha\beta - \gamma\beta^2}{V\alpha\beta} - x\beta^2\right] \\
&= \alpha\left[-S\frac{\beta}{\alpha}S\frac{\gamma}{V\alpha\beta} + S\frac{\delta}{V\alpha\beta}\right] \\
&\quad - \beta\left[-S\frac{\gamma}{V\alpha\beta} + S\frac{\alpha}{\beta}S\frac{\delta}{V\alpha\beta}\right] \\
&= \alpha^{-1}\left[-S\alpha\beta S\frac{\gamma}{V\alpha\beta} + \alpha^2 S\frac{\delta}{V\alpha\beta}\right] \\
&\quad - \beta^{-1}\left[-\beta^2 S\frac{\gamma}{V\alpha\beta} + S\alpha\beta S\frac{\delta}{V\alpha\beta}\right].
\end{aligned}
$$

Hence the vector perpendicular from the first line to the second is

$$-(V\alpha\beta)^{-1}(S\delta\alpha + S\beta\gamma)$$

and vectors to the intersections of this perpendicular with the first and second lines are respectively

$$\gamma\alpha^{-1} - \alpha^{-1}\left[S\frac{\gamma S\alpha\beta - \delta\alpha^2}{V\alpha\beta}\right]$$

and

$$\delta\beta^{-1} + \beta^{-1}\left[S\frac{\delta S\alpha\beta - \gamma\beta^2}{V\alpha\beta}\right].$$

* Note that

$$
\begin{aligned}
(V\alpha\beta)^{-1}V(V\alpha\beta)(x\beta - y\alpha) &= x\beta - y\alpha \\
(V\alpha\beta)^{-1}V\cdot V\alpha\beta(\delta\beta^{-1} - \gamma\alpha^{-1}) &= (V\alpha\beta)^{-1}(-\alpha^{-1}S\beta\gamma\alpha - \beta^{-1}S\alpha\delta\beta) \\
&= V\alpha\beta\left(-\alpha^{-1}S\frac{\gamma}{V\alpha\beta} + \beta^{-1}S\frac{\delta}{V\alpha\beta}\right).
\end{aligned}
$$

The projections of the vectors α, γ on any three rectangular axes give the Pluecker coordinates of the line. For applications to linear complexes, etc., see Joly: Manual, p. 40, Guiot: Le Calcul Vectoriel et ses applications.

2. **Congruence.** The differential equation of a curve or set of curves forming a congruence whose tangents have given directions σ, that is, the vector lines of a vector field σ, is given by

$$Vd\rho\sigma = 0$$

or its equivalent equation

$$d\rho = \sigma dt.$$

3. **Moment.** The moment of the force β with respect to a point whose vector from an origin on the line of β is α, is $-V\alpha\beta$. If the point is the origin and the vector to some point in the line of application of the force is α, then the moment with respect to the origin is $V\alpha\beta$. If the point is on the line of application the moment obviously vanishes. If several forces have a common plane then the moments as to a point in the plane will have a common unit vector, the normal to the plane. If several forces are normal to the same plane, their points of application in the plane given by β_1, β_2, β_3, $\cdots$, their values being $a_1\alpha$, $a_2\alpha$, $a_3\alpha$, $\cdots$, then the moments are

$$V(a_1\beta_1 + a_2\beta_2 + a_3\beta_3 + \cdots)\alpha \qquad \text{[dyne cm.]}.$$

If we set

$$a_1\beta_1 + a_2\beta_2 + a_3\beta_3 + \cdots = \beta(a_1 + a_2 + a_3 + \cdots),$$

then β is the vector to the *mean point* of application, which, in case the forces are the attractions of the earth upon a set of weighted points, is called the center of gravity. If $a_1 + a_2 + a_3 + \cdots = 0$, we cannot make this substitution.

4. **Couple.** A couple consists of two forces of equal magnitude, opposite directions and different lines of action. In such case the mean point becomes illusory and the sum of the moments for any point from which vectors to points on the lines of action of the forces are α_1, α_2 respectively, is

$$V(\alpha_1 - \alpha_2)\beta.$$

But $\alpha_1 - \alpha_2$ is a vector from one line of action to the other, and this sum of the moments is called the moment of the couple. It is evidently unchanged if the tensor of β is increased and that of $\alpha_1 - \alpha_2$ decreased in the same ratio, or vice versa.

5. **Moment of Momentum.** If the velocity of a moving mass m is σ cm./sec., then the momentum of the mass is defined to be $m\sigma$ gr. cm./sec. The vector to the mass being ρ, the moment of momentum of the mass is defined to be

$$V\rho m\sigma = mV\rho\sigma \qquad [\text{gm. cm.}^2/\text{sec.}].$$

6. **Electric Intensity.** If a medium is moving in a magnetic field of density $\mathbf{B}$ gausses, with a velocity σ cm./sec., then there will be set up in the medium an electromotive intensity $\mathbf{E}$ of value

$$\mathbf{E} = V\sigma\mathbf{B}\cdot 10^{-8} \qquad [\text{volts/centimeter}].$$

For any path the volts will be

$$-\int Sd\rho\mathbf{E} = +\int Sd\rho\mathbf{B}\sigma\cdot 10^{-8}.$$

If this be integrated around any complete circuit we shall arrive at the difference in electromotive force at the ends of the circuit.

7. **Magnetic Intensity.** If a magnetic medium is moving in an induction field of $\mathbf{D}$ lines, with a velocity σ, then there will be produced in the medium at every point a magnetic

intensity field

$$\mathbf{H} = 0.4\pi V\mathbf{D}\sigma \qquad \text{[gilberts/cm.]}.$$

For any path the gilbertage will be $0.4\pi\int Sd\rho\sigma\mathbf{D}$.

8. **Moving Electric Field.** If an electric field of induction, of value $\mathbf{D}$ lines, is moving with a velocity σ, then there will be produced in the medium at the point a magnetic field of intensity $\mathbf{H}$ gilberts/cm. where

$$\mathbf{H} = 0.4\pi V\sigma\mathbf{D}.$$

For a moving electron with charge e, this will be $-(eU\rho/4\pi T\rho^2)$. For a continuous stream of electrons along a path we would have

$$\mathbf{H} = \int_A^B V\cdot\left(\frac{eU\rho}{40\pi T\rho^2}\cdot d\rho\right) = \frac{e}{40\pi}\int_A^B V\cdot d\rho\nabla\frac{1}{Tp},$$

the point being the origin.

9. **Moving Magnetic Field.** If a magnetic field of induction of value $\mathbf{B}$ gausses is moving with a velocity σ, it will produce at any given point in space an electric intensity $\mathbf{E} = V\cdot\mathbf{B}\sigma 10^{-8}$ volts per centimeter.

10. **Torque.** If a particle of length $d\rho$ is in a field of intensity σ which tends to turn the particle along the lines of force, then the torque produced by the field upon the element is

$$V\cdot d\rho\sigma.$$

If a line runs from A to B, the total torque is

$$\int_A^B V\cdot d\rho\sigma.$$

For instance if $d\rho$, or in case of a non-uniform distribution $cd\rho$, is the strength in magnetic units, maxwells, of a wire magnet from A to B, in a field σ, then

$$\int_A^B V\cdot d\rho\sigma \qquad \text{or} \qquad \int_A^B V\cdot cd\rho\sigma$$

is the torque of the field upon the magnet.

11. **Poynting Vector.** An electric intensity $\mathbf{E}$ volts/cm. and magnetic intensity $\mathbf{H}$ gilberts/cm. at a point in space are accompanied by a flux of energy per cm.^{2} $\mathbf{R}$, given by the formula

$$4\pi\mathbf{R} = \frac{V\mathbf{EH}}{300} \qquad [\text{ergs/cm.}^2\ \text{sec.}].$$

This is the Poynting vector.

12. **Force Density.** The force density in dynes/cc. of a field of electric induction on a magnetic current is given by,

$$\mathbf{F} = 4\pi V\mathbf{DG} : 10 \qquad [\text{dynes/cc.}],$$

where $\mathbf{D}$ is the density in lines of electric displacement $\mathbf{G}$ is the magnetic current density in heavisides per cm.^{2}. If the negative of $\mathbf{F}$ is considered we have the force per cc. required to hold a magnetic current in an electrostatic field of density $\mathbf{D}$.

The force density in dynes/cc. of a field of magnetic induction on a conductor carrying an electric current is

$$\mathbf{F} = \frac{1}{10} V \cdot \mathbf{JB}.$$

A single moving charge e with velocity σ will give

$$\mathbf{F} = .4\pi e V\sigma\mu V\sigma\mathbf{D}.$$

13. **Momentum of Field.** The field momentum at a point where the electric induction is $\mathbf{D}$ lines and magnetic induction $\mathbf{B}$ gausses is $\Gamma = 3 \cdot 10^9\ V \cdot \mathbf{DB}$ [gm. cm./sec.]. If the magnetic induction is due to a moving electric field then $\Gamma = 0.04\pi V \cdot \mathbf{D}\mu V\mathbf{D}\sigma$, and if the electric induction is due to a moving magnetic field,

$$\Gamma = \frac{1}{4\pi \cdot 3 \cdot 10^{10}} V\mathbf{B}\kappa V\sigma\mathbf{B}.$$

3. The Scalar of Three Vectors

1. **Area and Pressure.** If we consider two differential vectors from the point P, say $d_1\rho$, $d_2\rho$, then the vector area of the parallelogram they form is $Vd_1\rho d_2\rho$. If then we have a distribution of an areal character, such as pressure per square centimeter, β, the pressure normal to the differential area will be in magnitude

$$-S\beta d_1\rho d_2\rho.$$

The vector $Vd_1\rho d_2\rho$ may be represented by $d\nu$ or $U\nu dA$.

The vector pressure normal to the surface will be

$$U\nu S\beta d_1\rho d_2\rho.$$

There will also be a tangential pressure or shear, which is the other component of β.

2. **Flux.** If β is any vector distribution the expression $-S\beta d_1\rho d_2\rho$ is often called the flux of β through the area $Vd_1\rho d_2\rho$. It is to be noted however that the dimensions of the result in physical units must be carefully considered. Thus the flux of magnetic intensity is of dimensions that do not correspond to any magnetic quantity.

3. **Flow.** If β is the velocity of a fluid in cm./sec., then the volume passing through the differential area per second is

$$-S\beta d_1\rho d_2\rho \qquad \text{[cc./sec.]}.$$

4. **Energy Flux.** The dimensions of the Poynting energy flow $\mathbf{R}$ show that it is the current of energy per second across a cm.^{2}, hence the total flow per second through an area is

$$-S\mathbf{R}d_1\rho d_2\rho = -\frac{S\cdot V\mathbf{EH}Vd\rho_1 d\rho_2}{1200\pi} \qquad \text{[ergs/sec.]}.$$

In the case of a straight conductor carrying a current of electricity, we have at a distance a from the wire in a

direction at right angles to the wire directly away from it the value

$$T \cdot \mathbf{R} = (4\pi)^{-1}10^8 E(0.2Ja^{-1}).$$

Consequently if we consider one centimeter of wire in length and the circumference of the circle of radius a we shall have a flux of energy for the centimeter equal to

$$\mathbf{J}(E_2 - E_1) \qquad \text{[joules]}.$$

This is the usual J^2R of a wire and is represented by heat.

5. **Activity.** For a moving conductor we have already expressed the vector $\mathbf{E}$, and as the current density $\mathbf{J}$ can be computed from the intensity of the field ($\mathbf{J} = k\mathbf{E}$) we have then for the expression of the activity in watts per cubic centimeter of conductor

$$A = -S\sigma\mathbf{BJ}10^{-8} = -S(V\sigma\mathbf{B})k(V\sigma\mathbf{B})\cdot 10^{-16} \quad \text{[watts]}.$$

Likewise in the case of the magnetomotive force due to motion and the magnetic current $\mathbf{G} = l\mathbf{H}$ we have for the activity per cubic centimeter of circuit

$$A = -S\mathbf{D}\sigma\mathbf{G} = -S\cdot(V\mathbf{D}\sigma)l(V\mathbf{D}\sigma)\cdot 10^{-7} \quad \text{[watts]}.$$

6. **Volts.** The total electromotive difference between two points in a conductor is the line-integral along the conductor

$$-\int Sd\rho\sigma\mathbf{B}10^{-8} \qquad \text{[volts]}.$$

7. **Gilberts.** The total magnetomotive difference between two points along a certain path is the line-integral

$$-.4\pi\int Sd\rho\mathbf{D}\sigma \qquad \text{[gilberts]}.$$

4. Vector of Three Vectors

1. **Stress.** We find with no difficulty the equations

$$V\cdot\alpha(U\alpha \pm U\gamma)\gamma = \pm T\gamma T\alpha(U\alpha \pm U\gamma)$$

and

$$V\cdot\alpha(V\alpha\gamma)\gamma = -S\alpha\gamma\cdot V\cdot\alpha\gamma.$$

If now we have a state of stress in a medium, given by its three principal stresses in the form

$g_1 = g - T\lambda\mu$ dynes/cm.^{2} normal to the plane orthogonal to $U(U\lambda + U\mu)$,
$g_2 = g - S\lambda\mu$ dynes/cm.^{2} normal to the plane orthogonal to $UV\lambda\mu$,
$g_3 = g + T\lambda\mu$ dynes/cm.^{2} normal to the plane orthogonal to $U(U\lambda - U\mu)$,

$$g_1 < g_2 < g_3,$$

then the stress across the plane normal to β is

$$V\lambda\beta\mu + g\beta.$$

If the scalars g_1, g_2, g_3 are dielectric constants in three directions (trirectangular) properly chosen, then the displacement is

$$\mathbf{D} = V\lambda\mathbf{E}\mu + g\mathbf{E}.$$

If the scalars are magnetic permeability constants,

$$\mathbf{B} = V\lambda\mathbf{H}\mu + g\mathbf{H}.$$

If the scalars are coefficients of dilatation, then β becomes

$$\beta' = V\lambda\beta\mu + g\beta.$$

If the scalars are elasticity constants of the ether, then according to Fresnel's theory, the force on the ether is, for the ether displacement β,

$$V\lambda\beta\mu + g\beta.$$

If the scalars are thermoelectric constants in a crystal, then

$$\mathbf{D} = V\lambda\mathbf{Q}\mu + g\mathbf{Q} \qquad \text{where } \mathbf{Q} \text{ is the flow of heat.}$$

If $g = 0$ the scalars are $T\lambda\mu$, $-T\lambda\mu$, $-S\lambda\mu$. If $V\lambda\mu = 0$, the scalars are $T\lambda\mu$, $-T\lambda\mu$, $T\lambda\mu$, that is, practically $-t$ along λ and $+t$ in all directions perpendicular to λ.

CHAPTER VIII

DIFFERENTIALS AND INTEGRALS

1. Differentiation as to a Scalar Parameter

1. **Differential of ρ.** If the vector ρ depends upon the scalar parameter t, say

$$\rho = \varphi(t),$$

then for two values of t which are supposed to be in the range of possible values for t

$$\frac{\rho_2 - \rho_1}{t_2 - t_1} = \frac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1}.$$

If now we suppose that $t_0 < t_1 < t_2$ and that t_1 and t_2 can independently approach the limit, t_0, then we shall call the limit of the fraction above, if there be such a limit, the right-hand derivative of ρ as to t, at t_0, and if $t_2 < t_1 < t_0$, we shall call the limit the left-hand derivative of ρ as to t at t_0. In case these both exist and are equal, and if ρ has a value for t_0 which is the limit of the two values of $\varphi(t_1)$, then we shall say that ρ is a continuous function of t at t_0 and has a derivative as to t at t_0.

There is no essential difference analytically between the function φ and the ordinary functions of a single real variable, and we will assume the ordinary theory as known.

It is evident that for different values of t we may consider the locus of P which will be a continuous curve. Since $\rho_2 - \rho_1$ is a chord of the curve the limit above will give a vector along the tangent of the curve. Further the tensor of the derivative, $T\rho' = T\varphi'(t)$, is the derivative of the length of the arc as to the parameter t. If the arc s is the parameter then the vector ρ' is a unit vector.

EXAMPLES

(1) The circle

$$\rho = \alpha \cos \theta + \beta \sin \theta, \qquad T\alpha = T\beta, \qquad S\alpha\beta = 0,$$
$$\rho' = -\alpha \sin \theta + \beta \cos \theta.$$

(2) The helix

$$\rho = \alpha \cos \theta + \beta \sin \theta + \gamma\theta,$$
$$\rho' = -\alpha \sin \theta + \beta \cos \theta + \gamma.$$

(3) The conic

$$\rho = \frac{\alpha t^2 + 2\beta t + \gamma}{a t^2 + 2bt + c}.$$

Multiplying out, $t^2(\alpha - a\rho) + 2t(\beta - b\rho) + (\gamma - c\rho) = 0$ for all values of t. For $t = 0$, $\rho = \gamma/c$, and for $t = \infty$, $\rho = \alpha/a$, hence the curve goes through α/a and γ/c.

We have

$$d\rho/dt = [t^2(b\alpha - a\beta) + t(c\alpha - a\gamma) + (c\beta - b\gamma)] \text{ times scalar.}$$

Hence for $t = 0$, the direction of the tangent is $\beta/b - \gamma/c$ at γ/c, for $t = \infty$, the direction of the tangent is $\beta/b - \alpha/a$ at α/a. Since these vectors both run from the points of tangency to the point β/b, the curve is a conic, tangent to the lines through β/b and the two points α/a and γ/c, at these two points. If the origin is taken at β/b, so that $\rho = \pi + \beta/b$, and if $\alpha' = \alpha/a - \beta/b$, $\gamma' = \gamma/c - \beta/b$, then

$$at^2(\alpha' - \pi) - 2bt\pi + c(\gamma' - \pi) = 0$$

is the equation of the curve.

If now we let π run along the diagonal of the parallelogram whose two sides are $\alpha'\gamma'$ so that $\pi = x(\alpha' + \gamma')$, then substituting we have

$$at^2x + 2btx - c(1 - x) = 0,$$
$$at^2(1 - x) - 2btx - cx = 0.$$

From these equations we have

$$t^2 = c/a$$

and

$$x = \sqrt{ac}/2(\sqrt{ac} \pm b).$$

These values of x give us the two points in which the diagonal in question cuts the curve. The middle point between these two is

$$\tau = \tfrac{1}{2}(x_1 + x_2)(\alpha' + \gamma') = \frac{bc\alpha + ab\gamma - 2ca\beta}{2b(ac - b^2)}.$$

Referred to the original origin this gives for the center

$$\kappa = \tau + \beta/b = \frac{c\alpha - 2b\beta + a\gamma}{2(ac - b^2)}.$$

If we calculate the point on the curve for

$$t_2 = -\frac{bt_1 + c}{at_1 + b},$$

we shall find that for the points ρ_2, ρ_1 we have $\frac{1}{2}(\rho_2 + \rho_1) = \kappa$, so that κ is the center of the curve and diametrically opposite points have parameters

$$t_1 \text{ and } t_2 = \frac{bt_1 + c}{at_1 + b},$$

an involutory substitution. If $ac = b^2$, κ becomes ∞ except when also the numerator $= 0$. [Joly, Manual, Chap. VII, art. 48.]

In general the equation of the tangent of any curve is

$$\pi = \rho + x\rho'.$$

We may also find the derivatives of functions of ρ, when $\rho = \varphi(t)$, by substituting the value of ρ in the expression and differentiating as before. Thus

$$\text{let } \rho = \alpha \cos\theta + \beta \sin\theta \qquad \text{where } T\alpha \mp T\beta.$$

Then

$$T\rho = \sqrt{[-\alpha^2 \cos^2\theta - 2S\alpha\beta \sin\theta\cos\theta - \beta^2\sin^2\theta]}.$$

We may then find the stationary values of $T\rho$ in the manner usual for any function. Thus differentiating after squaring

$$\alpha^2 \sin 2\theta - 2S\alpha\beta \cos 2\theta - \beta^2 \sin 2\theta = 0,$$
$$\tan 2\theta = 2S\alpha\beta/(\alpha^2 - \beta^2).$$

2. **Frenet-Serret Formulae.** Since the arc is essentially the natural parameter of a curve we will suppose now that ρ is expressed in terms of s, and accents will mean only differentiation as to s. Then both

$$\rho \qquad \text{and} \qquad \rho + ds\rho'$$

are points upon the curve.

The derivative of the latter gives $\rho' + ds\rho''$, which is also a unit vector since the parameter is s. Thus the change in a unit vector along the tangent is $ds\rho''$, and since this vector is a chord of a unit circle its limiting direction is perpendicular to ρ', and its quotient by ds has a length whose limit is the rate of change of the angle in the osculating plane of the tangent and a fixed direction in that plane which turns with the plane. That is to say, ρ'' in direction is along the principal normal of the curve on the concave side, and in magnitude is the *curvature* of the curve, which we shall indicate by the notation

Unit tangent is $\alpha = \rho'$,
Unit normal is $\beta = U\rho''$, curvature is $c_1 = T\rho''$,
Unit binormal is $\gamma = V\alpha\beta$, so that $c_1\gamma = V\rho'\rho''$.

The rate of angular turn of the osculating plane per centimeter of arc is found by differentiating the unit normal of the plane. Thus we have

$$\gamma_1 \doteq c_1^{-2}[c_1 V\rho'\rho''' - V\rho'\rho''\cdot c_2].$$

But $c_1^2 = T^2\rho'' = - S\rho''\rho''$ and therefore $c_1c_2 = - S\rho''\rho'''$. Substituting for c_2 we have

$$\begin{aligned}\gamma_1 &= c_1^{-3}[- S\rho''\rho''V\rho'\rho''' + S\rho''\rho'''V\rho'\rho''] \\ &= c_1^{-3}[V\rho'V\rho''V\rho'''\rho''] \\ &= c_1^{-3}V\alpha Vc_1\beta V\rho'''c_1\beta \\ &= c_1^{-1}V\cdot\alpha\beta V\rho'''\beta = c_1^{-1}V\gamma V\rho'''\beta = c_1^{-1}\beta S\gamma\rho''' \\ &= - a_1\beta,\end{aligned}$$

where a_1 is written for the negative tensor of γ_1 and is the *tortuosity*. It may also be written in the form

$$S\rho'\rho''\rho'''/c_1^2.$$

Again since $\beta = \gamma\alpha$ we have at once the relations

$$\beta_1 = \gamma_1\alpha + \gamma\alpha_1 = a_1\gamma - c_1\alpha.$$

Thus we have proved *Frenet's* formulae for any curve

$$\alpha_1 = c_1\beta, \qquad \beta_1 = a_1\gamma - c_1\alpha, \qquad \gamma_1 = - a_1\beta.$$

It is obvious now that we may express derivatives of any order in terms of α, β, γ, and a_1, c_1, and the derivatives of a_1 and c_1.

For instance we have

$$\begin{gathered}\rho_1 = \alpha, \qquad \rho_2 = \beta c_1, \\ \rho_3 = \beta_1c_1 + \beta c_2 = \beta c_2 + (\gamma a_1 - \alpha c_1)c_1, \\ \rho_4 = \beta c_3 + 2(\gamma a_1 - \alpha c_1)c_2 + (\gamma a_2 - \alpha c_2)c_1 \\ - \beta(a_1^2 + c_1^2)c_1.\end{gathered}$$

. . . .

The vector $\omega = \alpha a_1 + \gamma c_1$ is useful, for if η represents in turn each one of the vectors α, β, γ, then $\eta_1 = V\omega\eta$. It is the vector along the rectifying line through the point.

The centre of absolute curvature κ is given by

$$\kappa = \rho - 1/\rho'' = \rho + \beta/c_1.$$

The centre of spherical curvature is given by

$$\sigma = \kappa + \gamma d/da \cdot c_1^{-1} = \kappa - \gamma c_2/a_1 c_1^2.$$

The polar line is the line through K in the direction of γ. It is the ultimate intersection of the normal planes.

3. **Developables.** If we desire to study certain developables belonging to the curve, a developable being the locus of intersections of a succession of planes, we proceed thus. The equation of a plane being $S(\pi - \rho)\eta = 0$, where π is the vector to a variable point of the plane, and ρ is a point on the curve, while η is any vector belonging to the curve, then the consecutive plane is

$$S(\pi - \rho)\eta + ds \cdot \partial/\partial s S(\pi - \rho)\eta = 0.$$

The intersection of this and the preceding plane is the line whose equation is

$$\pi = \rho + (-\eta S\alpha\eta + t)/V\eta\eta_1.$$

This line lies wholly upon the developable. If we find a second consecutive plane the intersection of all three is a point upon the cuspidal edge of the developable, which is also the locus of tangents of the cuspidal edge. This vector is

$$\pi = \rho + (V\eta_2\eta S\alpha\eta + 2V\eta\eta_1 S\alpha\eta_1 + V\eta\eta_1 S\beta\eta c_1)/S\eta\eta_1\eta_2.$$

By substituting respectively for η, α, β, γ, we arrive at the polar developable, the rectifying developable, the tangent-line developable.

EXAMPLE

Perform the substitutions mentioned.

4. **Trajectories.** If a curve be looked upon as the path of a moving point, that is, as a trajectory, then the parameter becomes the time. In this case we find that (if $\dot{\rho} = d\rho/dt$, etc.) the velocity is $\dot{\rho} = \alpha v$, the acceleration is

$\ddot{\rho} = \beta c_1 v^2 + \alpha \dot{v}$. The first term is the acceleration normal to the curve, the centrifugal force, the second term is the tangential acceleration. In case a particle is forced to describe a curve, the pressure upon the curve is given by $\beta c_1 v^2$. There will be a second acceleration, $\dddot{\rho} = \alpha(\ddot{v} - v c_1{}^2) + \beta(2c_1\dot{v} + c_2 v) + \gamma a_1 c_1 v$. The last term represents a tendency per gram to draw the particle out of the osculating plane, that is, to rotate the plane of the orbit.

5. **Expansion for ρ.** If we take a point on the curve as origin, we may express ρ in the form

$$\rho = s\alpha + \tfrac{1}{2}c_1 s^2\beta - \tfrac{1}{6}s^3(c_1{}^2\alpha - c_2\beta - c_1 a_1\gamma)$$
$$- \tfrac{1}{24}s^4(3c_2c_1\alpha - \beta[c_3 - c_1{}^3 - c_1 a_1{}^2] - \gamma[2c_2a_1 + c_1a_2]) \cdots.$$

EXERCISES

1. Every curve whose two curvatures are always in a constant ratio is a cylindrical helix.

2. The straight line is the only real curve of zero curvature everywhere.

3. If the principal normals of a curve are everywhere parallel to a fixed plane it is a cylindrical helix.

4. The curve for which

$$c_1 = 1/ms, \qquad a_1 = 1/ns,$$

is a helix on a circular cone, which cuts the elements of the cone under a constant angle.

5. The principal normal to a curve is normal to the locus of the centers of curvature at points where c_1 is a maximum or minimum.

6. Show that if a curve lies upon a sphere, then

$c_1{}^{-1} = A \cos a + B \sin a = C \cos (a + \epsilon)$, A, B, C, ϵ are constants. The converse is also true.

7. The binormals of a curve do not generate the tangent surface of a curve.

8. Find the conditions that the unit vectors of the moving trihedral $\alpha\beta\gamma$ of a given curve remain at fixed angles to the unit vectors of the moving trihedral of another given curve.

Two Parameters

6. **Surfaces.** If the variable vector ρ depends upon two arbitrary parameters it will terminate upon a surface of

some kind. For instance if $\rho = \varphi(u, v)$, then we may write for the total differential of ρ

$$d\rho = du\partial/\partial u(\varphi) + dv\partial/\partial v(\varphi) = du\varphi_u + dv\varphi_v.$$

We find then

$$T^2 d\rho = Edu^2 + 2Fdudv + Gdv^2,$$

where

$$E = -\varphi_u^2, \qquad F = -S\varphi_u\varphi_v, \qquad G = -\varphi_v^2.$$

We have thus two differentials of ρ, one for $v =$ constant, one for $u =$ constant, which will be tangent to the parametric curves upon the surface given by these equations, and may be designated by

$$\rho_1 du, \qquad \rho_2 dv.$$

The normal becomes then

$$\nu = V\rho_1\rho_2, \qquad T\nu = \sqrt{(EG - F^2)} = H.$$

For certain points or lines ν may become indeterminate, the points or lines being then singular points or singular lines.

7. Curvatures. If we consider the point ρ and the point $\rho + du\rho_1 + dv\rho_2$ the two normals will be

$$\nu \quad \text{and} \quad \nu + duV(\rho_{11}\rho_2 + \rho_1\rho_{12}) + dvV(\rho_{12}\rho_2 + \rho_1\rho_{22}) + \cdots$$

which may be written

$$\nu \quad \text{and} \quad \nu + d\nu.$$

The equations of these lines are

$$V(\pi - \rho)\nu = 0, \qquad V(\pi - \rho - d\rho)(\nu + d\nu) = 0.$$

They intersect if

$$Sd\rho\nu d\nu = 0.$$

Points for which this equation holds lie upon a line of

curvature so that this is the differential equation of such lines. If we expand the total differentials we have

$$du^2S\rho_1\nu\nu_1 + 2dudvS(\rho_1\nu\nu_2 + \rho_2\nu\nu_1) + dv^2S\rho_2\nu\nu_2 = 0.$$

We may also write the equation in the form

$$d\rho + x\nu + yd\nu = 0 = \rho_1du + \rho_2dv + x\nu + y\nu_1du + y\nu_2dv.$$

Multiply by $(\rho_1 + y\nu_1)(\rho_2 + y\nu_2)$ and take the scalar part of the product, giving

$$S(\rho_1 + y\nu_1)(\rho_2 + \nu_2y)\nu = 0$$
$$= y^2S\nu\nu_1\nu_2 + 2yS\nu(\rho_1\nu_2 + \nu_1\rho_2) + \nu^2.$$

The ultimate intersection of the two normals is given by

$$\pi = \rho + d\rho + y\nu + yd\nu,$$

that is by $y\nu$. Hence we solve for $yT\nu$, giving two values R and R' which are the principal radii of curvature at the point. The product and the sum of the roots are respectively

$$RR' = yy'T\nu^2 = T\nu^4/(-S\nu\nu_1\nu_2),$$
$$R + R' = -2T\nu S\nu(\rho_1\nu_2 + \nu_1\rho_2)/S\nu\nu_1\nu_2.$$

The reciprocal of the first, and one-half the second divided by the first, that is,

$$-S\nu\nu_1\nu_2/\nu^4 \qquad \text{and} \qquad S\nu(\rho_1\nu_2 + \nu_1\rho_2)/T\nu^3,$$

are the absolute curvature and the mean curvature of the surface at the point.

The equation of the lines of curvature may be also written

$$\nu Sd\rho\nu d\nu = 0 = V\cdot Vd\rho V\nu d\nu = Vd\rho V(d\nu/\nu\cdot\nu) = Vd\rho dU\nu.$$

Hence the direction of $dU\nu$ is that of a line of curvature, when du and dv are chosen so that $d\rho$ follows the line of curvature. That is, along a line of curvature the change

in the direction of the unit normal is parallel to the line of curvature.

When the mean curvature vanishes the surface is a minimal surface, the kind of surface that a soapfilm will take when it extends from one curve to another and the pressures on the two sides are equal. The pressure indeed is the product of the surface tension and twice the mean curvature, so that if the resultant pressure is zero, the mean curvature must vanish. If the radii are equal, as in a sphere, then the resultant pressure will be twice the surface tension divided by the radius, for each surface of the film, giving difference of pressure and air pressure = 4 times surface tension/radius. The difference of pressure is thus for a sphere of 4 cm. radius equal to the surface tension, that is, 27.45 dynes per cm.

When a surface is developable the absolute curvature is zero, and conversely. Surfaces are said to have positive or negative curvature according as the absolute curvature is positive or negative.

EXERCISES

1. The differential equation of spheres is

$$V\nu(\rho - \alpha) = 0.$$

2. The differential equations of cylinders and cones are respectively

$$S\nu\alpha = 0, \qquad S\nu(\rho - \alpha) = 0.$$

3. The differential equation of a surface of revolution is

$$S\alpha\rho\nu = 0.$$

4. Why is the center of spherical curvature of a spherical curve not of necessity the center of the sphere?

5. Show how to find the vector to an umbilicus (the radii of curvature are equal at an umbilicus).

6. The differential equation of surfaces generated by lines that are perpendicular to the fixed line α is

$$SV\alpha\nu\varphi V\alpha\nu = 0,$$

where φ is a linear function.

7. The differential equation of surfaces generated by lines that meet the fixed line $V(\rho - \beta)\alpha = 0$ is

$$SV\nu V(\rho - \beta)\alpha\varphi(V\nu V(\rho - \beta)\alpha) = 0.$$

8. The differential equation of surfaces generated by equal and similarly situated ellipses is

$$SV(V\alpha\beta \cdot \nu)\varphi(VV\alpha\beta \cdot \nu) = 0.$$

9. Show that the catenoid

$$\rho = xi + a \cosh x/a(\cos \theta j + \sin \theta k)$$

is a minimal surface, and that the two radii are $\pm T\nu$, the normal which is drawn from the point to the axis.

2. Differentiation as to a Vector

1. **Definition.** Let $q = f(\rho)$ be a function of ρ, either scalar, vector, or quaternion. Let ρ be changed to $\rho + dt \cdot \alpha$ where α is a unit vector, then the change in q is given by

$$dq = q' - q = f(\rho + dt \cdot \alpha) - f(\rho),$$

and

$$dq/dt = \text{Lim } [f(\rho + dt\alpha) - f(\rho)]/dt$$

as dt decreases. If we consider only the terms in first order of the infinitesimal scalar dt we can write

$$dq = dtf'(\rho, \alpha)$$

in which α will enter only linearly.

In a linear function of α however we can introduce the multiplier into every term in α and write $dt\alpha = d\rho$, so that we have dq a linear function of $d\rho$,

$$dq = f'(\rho, d\rho).$$

It needs to be noted that the vector α is a function of the variable dt, although a unit vector. The differential of q is of course a function of the direction of $d\rho$ in general, but the direction may be arbitrary, or be a function of the variable vector ρ. It may very well happen that the limit obtained above may be different for a given function f according to the direction of the vector α. In general, we intend to consider the

vector $d\rho$ as having a purely arbitrary direction unless the contrary is stated.

EXAMPLES

(1) Let

$$q = -\rho^2.$$

Then

$$dq = -[\rho^2 + 2dtS\cdot\rho\alpha - \rho^2] = -2dtS\rho\alpha = -2S\rho d\rho.$$

Also since $q = T^2\rho$ we have

$$dq = 2T\rho dT\rho = -2S\rho d\rho,$$

whence

$$dT\rho/T\rho = Sd\rho/\rho, \qquad \text{or} \qquad dT\rho = -SU\rho d\rho.$$

(2) From the definition we have

$$d(qr) = dq\cdot r + q\cdot dr,$$

hence

$$d(T\rho\cdot U\rho) = dT\rho\cdot U\rho + T\rho\cdot dU\rho = d\rho$$

and utilizing the result of the preceding example, we have

$$dU\rho/U\rho = Vd\rho/\rho.$$

Also we may write $dU\rho = -Vd\rho\rho\cdot\rho/T^3\rho = \rho Vd\rho\rho/T^3\rho = \rho^{-1}V\rho d\rho/T\rho$, etc. This equation asserts that the differential of $U\rho$ is the part of the arbitrary differential of ρ perpendicular to $U\rho$, divided by the length of ρ, that is, it is the differential angle of the two directions of ρ laid off in the direction perpendicular to ρ in the plane of ρ and $d\rho$. In case $d\rho$ is along the direction of ρ itself,

$$dU\rho = 0.$$

(3) We have since

$$d(\rho\rho^{-1}\rho) = d\rho = d\rho\cdot\rho^{-1}\rho + \rho d(\rho^{-1})\rho + \rho\rho^{-1}d\rho$$
$$= 2d\rho + \rho d(\rho^{-1})\rho,$$

and thence

$$d\rho = -\rho d(\rho^{-1})\rho,$$

$$d\cdot\rho^{-1} = -\rho^{-1}d\rho\rho^{-1} = [\rho^{-1}S\rho d\rho - \rho^{-1}V\rho d\rho]/T^2\rho$$
$$= \rho^{-1}d\rho\cdot\rho/T^2\rho.$$

That is, the differential of ρ^{-1} is the image of $d\rho$ in ρ divided by the square of $T\rho$.

Hence

$$d(V\alpha\rho)^{-1} = (V\alpha\rho)^{-1}V\alpha d\rho\cdot V\alpha\rho/T^2V\alpha\rho.$$

This vanishes if $d\rho$ is parallel to α.

(4) If $\pi = -a^2/\rho$ then $d\pi = -a^2\rho^{-1}d\rho\rho/T^2\rho$, and for two different values of $d\rho$, as $d_1\rho$, $d_2\rho$, we have

$$d_2\pi/d_1\pi = \rho^{-1}d_2\rho/d_1\rho\cdot\rho.$$

Therefore in the process of "inverting" or taking the "electrical image" we find that the biradial of two differential vectors is merely reflected in ρ. Interpret this.

(5) $T\dfrac{\rho+\alpha}{\rho-\alpha} = c$ is a family of spheres with α and $-\alpha$ as limit points. For a differential $d\rho$ confined to the surface of any sphere we have then

$$Sd\rho[(\rho+\alpha)^{-1} - (\rho-\alpha)^{-1}] = 0.$$

A plane section through α can be written $S\gamma\alpha\rho = 0$, in which $S\gamma\alpha d\rho = 0$ gives a differential confined to the plane. Therefore a differential tangent to the line of intersection of any plane and any sphere will satisfy the equation

$$Vd\rho[VV\gamma\alpha((\rho+\alpha)^{-1} - (\rho-\alpha)^{-1})] = 0.$$

But the expression in the () is a tangent line to any sphere which passes through A and $-A$. For the equation of such a sphere would be

$$\rho^2 - 2S\alpha\delta\rho - \alpha^2 = 0$$

where δ is any vector, hence for any $d\rho$ along the sphere, $S(\rho - V\alpha\delta)d\rho = 0$. But $(\rho + \alpha)^{-1} - (\rho - \alpha)^{-1}$ is parallel to $\alpha(\rho^2 + \alpha^2) - 2\rho S\alpha\rho$ and $S(\rho - V\alpha\delta)[\alpha(\rho^2 + \alpha^2) - 2\rho S\alpha\rho] = - S\alpha\rho[\rho^2 - \alpha^2 - 2S\rho\alpha\delta]$. For points on the sphere the [] vanishes, hence the vector in question is a tangent line. Also $V\pi\tau$ is perpendicular to π or τ, therefore the differential equation above shows that the tangent $d\rho$ of the intersection of the plane and the sphere of the system is perpendicular to a sphere through A and $-A$. Hence all spheres of the set cut orthogonally any sphere through A and $-A$.

(6) The equation $SU\dfrac{\rho + \alpha}{\rho - \alpha} = e$ is a family of tores produced by the rotation of a system of circles about their radical axis. From this we have

$$SU(\rho + \alpha)(\rho - \alpha) = - e,$$
$$VU(\rho + \alpha)(\rho - \alpha) = \sqrt{(1 - e^2)}UV\alpha\rho = \sigma.$$

Differentiating the scalar equation we have

$$S\left[V\frac{d\rho}{\rho + \alpha}\cdot U(\rho + \alpha)U(\rho - \alpha) + U(\rho + \alpha)V\frac{d\rho}{\rho - \alpha}\cdot U(\rho - \alpha)\right] = 0$$

or

$$S\sigma d\rho[(\rho + \alpha)^{-1} - (\rho - \alpha)^{-1}] = 0.$$

Now in a meridian section σ is constant so that

$$Vd\rho[(\rho + \alpha)^{-1} - (\rho - \alpha)^{-1}] = 0$$

and $d\rho$ is for such section tangent to a sphere through A and $-A$.

EXERCISES

1. The potential due to a mass m at the distance $T\rho$ is $m/T\rho$ in

gravitation units. Find the differential of the potential in any direction, and determine in what directions it is zero.

2. The magnetic force at the point P due to an infinite straight wire carrying a current σ is $\mathbf{H} = -2h/V\sigma\rho$. Find the differential of this and determine in what direction, if any, it is zero. For $Vd\rho\sigma = 0$, $d\mathbf{H} = 0$; for $d\rho = dsV\sigma^{-1}V\sigma\rho$, $d\mathbf{H} = -\mathbf{H}ds/TV\sigma\rho$; for $d\rho = dsUV\sigma\rho$, $d\mathbf{H} = V\sigma\mathbf{H}ds/TV\sigma\rho$.

3. The potential of a small magnet α at the origin on a particle of free magnetism at ρ is $u = S\alpha\rho/T^3\rho$. Find the variation in directions

$$U\rho, \qquad UV\alpha\rho, \qquad U\alpha V\alpha\rho.$$

4. The attraction of gravitation at a point P per unit mass in gravitation units is

$$\sigma = -U\rho/T^2\rho.$$

Find the differential of σ in the directions $U\rho$ and $V\beta\rho$.

$d\sigma = -(\rho^2 d\rho - 3\rho S\rho d\rho)/T^5\rho$; parallel to ρ, $-2/\rho^3$; perpendicular, $UV\beta\rho/T^3\rho$.

5. The force exerted upon a particle of magnetism at ρ by an element of current α at the origin is

$$\mathbf{H} = -V\alpha\rho/T^3\rho.$$

Then $d\mathbf{H} = (\rho^2 V\alpha d\rho - 3V\alpha\rho S\rho d\rho)/T^5\rho$; in the direction of ρ, $3V\alpha/\rho^3$; in the direction $V\alpha\rho$, $-V\alpha UV\alpha\rho/T^3\rho$.

6. The vector force exerted by an infinitesimal plane current at the origin perpendicular to α, upon a magnetic particle or pole at ρ is

$$\sigma = (\alpha\rho^2 - 3\rho S\alpha\rho)/T^5\rho.$$

Find its variation in various directions.

2. Differential of Quaternion. We may define differentials of functions of quaternions in the same manner as functions of vectors. Thus we have $T^2q = qKq$ so that

$$\begin{aligned} 2TqdTq = d(qKq) &= [(q + dtUq)(Kq + dtUKq) - qKq] \\ &= dt[qUKq + UqKq] \\ &= qKdq + dqKq \\ &= 2SqKdq = 2SdqKq. \end{aligned}$$

That is,

$$dTq = SdqUKq = SdqUq^{-1} = TqSdq/q$$

or

$$dTq/Tq = Sdq/q.$$

In the same manner we prove the other following formulae.

$$dUq/Uq = Vdq/q, \qquad dSq = Sdq, \qquad dVq = Vdq,$$
$$dKq = Kdq, \qquad S(dUq)/Uq = 0,$$
$$dSUq = SUqV(dq/q) = -S(dq/qUVq\cdot)TVUq$$
$$= TVUqd\angle q,$$
$$dVUq = VUKqV(dq/q),$$
$$dTVUq = -SdUqUVq = SUqd\angle q,$$
$$d\cdot q^2 = 2Sqdq + 2Sq\cdot Vdq + 2Sdq\cdot Vq,$$
$$d\cdot q^{-1} = -q^{-1}dqq^{-1},$$
$$d\cdot qaq^{-1} = -2V\cdot qdq^{-1}qVaq^{-1} = 2V\cdot dq(Va)q^{-1},$$

that is, if $r = qaq^{-1}$, then

$$dr = 2V(dq/q\cdot r) = -2V(q\cdot dq^{-1}\cdot r)$$
$$= 2V(Vdq/q)r = 2q\cdot V\cdot V(q^{-1}dq\cdot \alpha)q^{-1}$$
$$dUVq = V\cdot Vdq/Vq\cdot UVq,$$
$$d\angle q = S[dq/(UVq\cdot q)].$$

We define when $T\alpha = 1$

$$\alpha^x = \cos\cdot\pi x/2 + \sin\cdot\pi x/2\cdot\alpha = c\alpha s\cdot\tfrac{1}{2}\pi x;$$

thus

$$d\cdot\alpha^x = \pi/2\cdot\alpha^{x+1}dx.$$

If $T\alpha \neq 1$, then

$$d\cdot\alpha^x = dx[\log T\alpha\cdot\alpha^x + \pi/2\cdot\alpha^{x+1}/T\alpha],$$

$$d\sqrt{q} = \frac{dq}{4S\sqrt{q}} + \frac{1}{4S\sqrt{q}}\cdot\overline{q}^{1/2}dqKq^{1/2},*$$

$$d\cdot\rho^a = a\left(S\frac{d\rho}{\rho}\right)\cdot\rho^a + aV\frac{d\rho}{\rho}\cdot V\rho^a.$$

3. **Extremals.** For a stationary value of $f(\rho)$ in the vicinity of a point ρ we have $df(\rho) = 0$. If $f(\rho)$ is to be stationary and at the same time the terminal point of ρ is to remain on some surface, or in general if ρ is to be subject

* Tait, Quaternions, 3d ed., p. 97.

to certain conditioning equations, we must also have, if there is one equation, $q(\rho) = 0$, $dq(\rho) = 0$, and if there are two equations, $g(\rho) = 0$ and $h(\rho) = 0$, then also $dg(\rho) = 0$, $dh(\rho) = 0$. Whether in all these different cases $f(\rho)$ attains a maximum of numerical value or a minimum, or otherwise, we will consider later.

EXERCISES

1. $g(\rho) = (\rho - \alpha)^2 + a^2 = 0$, find stationary values of $T\rho = f(\rho)$. Differentiating both expressions,

$$Sd\rho(\rho - \alpha) = 0 = Sd\rho\rho,$$

for all values of $d\rho$. Hence we must have $d\rho$ parallel to $V\cdot\tau\rho$ where τ is arbitrary, and hence $S\tau\rho(\rho - \alpha) = 0$, for all values of τ. Therefore we must have $V\rho(\rho - \alpha) = 0$, or $V\alpha\rho = 0$, or $\rho = y\alpha$. Substituting and solving for y,

$$y = 1 \pm a/T\alpha, \qquad \rho = \alpha \pm aU\alpha.$$

2. $g(\rho) = (\rho - \alpha)^2 + a^2 = 0$. Find stationary values of $S\beta\rho$.

$Sd\rho(\rho - \alpha) = 0 = S\beta\alpha\rho$, whence $d\rho \| V\tau\beta$, $S\cdot\tau\beta(\rho - \alpha) = 0$, $V\beta(\rho - \alpha) = 0$.

$$\rho - \alpha = y\beta, \qquad y = a/T\beta, \qquad \rho = \alpha \pm aU\beta.$$

3. $g(\rho) = (\rho - \alpha)^2 + a^2 = 0$, $h(\rho) = S\beta\rho = 0$, find stationary values of $T\rho$.

$Sd\rho(\rho - \alpha) = 0 = S\beta d\rho = S\rho d\rho$, whence $S\cdot\rho\beta(\rho - \alpha) = 0 = S\rho\alpha\beta$, and since $S\rho\beta = 0$, $\rho = yV\cdot\beta V\alpha\beta$.

$$\rho = V\beta V\alpha\beta(1 \pm \sqrt{[a^2 - S^2\alpha\beta]}/TV\alpha\beta).$$

4. $g(\rho) = \rho^2 - S\alpha\rho S\beta\rho + a^2 = 0$. Find stationary values of $T\rho$.

$$Sd\rho\rho = 0 = Sd\rho(\rho - \alpha S\beta\rho - \beta S\alpha\rho),$$

$$\rho = x(\alpha S\beta\rho + \beta S\alpha\rho) = (\alpha S\beta\rho + \beta S\alpha\rho)/(S\alpha\beta \pm T\alpha\beta),$$

whence

$$S\alpha\rho = T\alpha S\rho U\beta,$$

$$\rho = S\rho U\beta(U\alpha \pm U\beta)/(SU\alpha U\beta \mp 1).$$

Substituting in the first equation, we find $S\rho U\beta$, thence ρ.

5. $S\beta\rho = c$, $S\alpha\rho = c'$, find stationary values of $T\rho$.

$$Sd\rho\beta = S\alpha d\rho = S\rho d\rho = 0, \qquad \rho = x\alpha + y\beta \qquad \text{and}$$

$$xS\alpha\beta + y\beta^2 = c, \qquad x\alpha^2 + yS\alpha\beta = c', \qquad \text{whence } x \text{ and } y.$$

6. Find stationary values of $S\alpha\rho$ when $(\rho - \alpha)^2 + a^2 = 0$.

$$S\alpha d\rho = 0 = Sd\rho(\rho - \alpha);$$

hence

$$\rho = y\alpha = \alpha \pm aU\alpha$$

and

$$S\alpha\rho = \alpha^2 \pm aT\alpha.$$

7. Find stationary values for $S\alpha\rho$ when $\rho^2 - S\beta\rho S\gamma\rho + a^2 = 0$.

$$S\alpha d\rho = 0 = Sd\rho(\rho - \beta S\gamma\rho - \gamma S\beta\rho),$$
$$\rho = x\alpha + \beta S\gamma\rho + \gamma S\beta\rho, \qquad \text{etc.}$$

8. Find stationary values of $TV\delta\rho$ when

$$(\rho - \alpha)^2 + a^2 = 0.$$

9. Find stationary values of $S\alpha U\rho$ when

$$(\rho - \alpha)^2 + a^2 = 0.$$

10. Find stationary values of $S\alpha U\rho S\beta U\rho$ when

$$S\gamma\rho + c = 0.$$

4. **Nabla.** The rate of variation in a given direction of a function of ρ is found by taking $d\rho$ in the given direction. Since $df(\rho)$ is linear in $d\rho$ it may always be written in the form

$$\Phi(d\rho),$$

where Φ is a linear quaternion, vector, or scalar function of $d\rho$. In case f is a scalar function, Φ takes the form

$$- Sd\rho\nu,$$

where ν is a function of ρ, which is usually independent of $d\rho$. In case ν is independent of the direction of $d\rho$, we call f a continuous, generally differentiable, function. Functions may be easily constructed for which ν varies with the direction of $d\rho$. If when ν is independent of $d\rho$ we take differentials in three directions which are not in the same plane, we have

$$\begin{aligned}\nu S\cdot d_1\rho d_2\rho d_3\rho &= V\cdot d_1\rho d_2\rho\cdot Sd_3\rho\nu + V\cdot d_2\rho d_3\rho\cdot Sd_1\rho\nu \\ &\qquad + V\cdot d_3\rho d_1\rho Sd_2\rho\nu \\ &= - V\cdot d_1\rho d_2\rho\cdot d_3 f - Vd_2\rho d_3\rho\cdot d_1 f \\ &\qquad - V\cdot d_3\rho d_1\rho\cdot d_2 f.\end{aligned}$$

It is evident that if we divide through by $Sd_1\rho d_2\rho d_3\rho$, the different terms will be differential coefficients. The entire expression may be looked upon as a differential operation upon f, which we will designate by ∇. Thus we have

$$\nu = \nabla f =$$
$$-\frac{(Vd_1\rho d_2\rho \cdot d_3 + V \cdot d_2\rho d_3\rho \cdot d_1 + V \cdot d_3\rho d_1\rho \cdot d_2)}{S \cdot d_1\rho d_2\rho d_3\rho} f(\rho).$$

We may then write

$$df(\rho) = -Sd\rho\nabla f(\rho).$$

If the three differentials are in three mutually rectangular directions, say i, j, k, then

$$\nabla = i\partial/\partial x + j\partial/\partial y + k\partial/\partial z.$$

It is easy to find ∇f for any scalar function which is generally differentiable from the equation for $df(\rho)$ above, that is, $df(\rho) = -Sd\rho\nabla f$. For instance,

$$\nabla S\alpha\rho = -\alpha, \qquad \nabla\rho^2 = -2\rho, \qquad \nabla T\rho = U\rho,$$
$$\nabla(T\rho)^n = nT\rho^{n-1}U\rho = nT\rho^{n-2}\cdot\rho, \qquad \nabla TV\alpha\rho = UV\alpha\rho\cdot\alpha,$$
$$\nabla S\alpha U\rho = -\rho^{-1}VU\rho\alpha, \qquad \nabla\cdot S\alpha\rho S\beta\rho = -\rho S\alpha\beta - V\alpha\rho\beta,$$

$$\nabla\cdot\log TV\alpha\rho = \frac{\alpha}{V\alpha\rho},$$

$$\nabla T(\rho - \alpha)^{-1} = -U(\rho - \alpha)/T^2(\rho - \alpha),$$
$$\nabla S\alpha U\rho S\beta U\rho = \rho^{-1}V\rho V\alpha\rho^{-1}\beta,$$
$$\nabla \log T\rho = U\rho/T\rho = -\rho^{-1},$$

$$\nabla(\angle\rho/\alpha) = -\rho^{-1}UV\rho\alpha = \frac{1}{\rho UV\alpha\rho}.$$

5. **Gradient.** If we consider the level surfaces of $f(\rho)$, $f(\rho) = C$, then we have generally for $d\rho$ on such surface or tangent to it $Sd\rho\mu = 0 = df(\rho)$ where μ is the normal of the

surface. Since $Sd\rho\nabla f = 0$ and since the two expressions hold for all values of $d\rho$ in a plane

$$\mu = x\nabla f,$$

or since the tensor of μ is arbitrary, we may say $\nabla f(\rho)$ is the normal to the level surface of $f(\rho)$ at ρ. It is called the *gradient of* $f(\rho)$, and by many authors, particularly in books on electricity and magnetism, is written *grad.* ρ.

The gradient is sometimes defined to be only the tensor of ∇f, and sometimes is taken as $-\nabla f$. Care must be exercised to ascertain the usage of each author.

Since the rate of change of $f(\rho)$ in the direction α is $-S\alpha\nabla f(\rho)$, it follows that the rate is a maximum for the direction that coincides with $U\nabla f$, hence the gradient

$$\nabla f(\rho)$$

gives the maximum rate of change of $f(\rho)$ *in direction and size.* That is, $T\nabla f$ is the maximum rate of change of $f(\rho)$ and $U\nabla f$ is the direction in which the point P must be moved in order that $f(\rho)$ shall have its maximum rate of change.

6. **Nabla Products.** The operator ∇ is sometimes called the *Hamiltonian* and it may be applied to vectors as well as to scalars, yielding very important expressions. These we shall have occasion to study at length farther on. It will be sufficient here to notice the effect of applying ∇ and its combinations to various expressions. It is to be observed that ∇Q may be found from dq, by writing $dq = \Phi\cdot d\rho$, then $\nabla Q = i\Phi i + j\Phi j + k\Phi k$.

For examples we have

$$\begin{aligned}\nabla\rho &= (Vd_1\rho d_2\rho\cdot d_3\rho + Vd_2\rho d_3\rho\cdot d_1\rho \\ &\qquad\qquad + Vd_3\rho d_1\rho\cdot d_2\rho)/(-Sd_1\rho d_2\rho d_3\rho) \\ &= -3\end{aligned}$$

since the vector part of the expression vanishes.

$$\nabla\rho^{-1} = -\,(Vd_1\rho d_2\rho\cdot\rho^{-1}d_3\rho\rho^{-1} + \cdots)/(-\,Sd_1\rho d_2\rho d_3\rho)$$
$$= -\,\rho^{-2}.$$

Since

$$dU\rho = V\frac{d\rho}{\rho}\cdot U\rho, \qquad dT\rho = -\,SU\rho d\rho.$$

Hence

$$\nabla U\rho = \Sigma iV\frac{i}{\rho}\cdot U\rho = -\,\frac{2}{T\rho}, \qquad \nabla T\rho = U\rho.$$

Expressions of the form $\Sigma F(i,\ i,\ Q)$ are often written $F(\zeta,\ \zeta,\ Q)$, a notation due to McAulay.

$$\nabla\alpha\rho = \alpha,$$

$$\nabla(\alpha_1S\beta_1\rho + \alpha_2S\beta_2\rho + \alpha_3S\beta_3\rho) = -\,(\beta_1\alpha_1 + \beta_2\alpha_2 + \beta_3\alpha_3),$$
$$\nabla V\alpha\rho = 2\alpha, \qquad \nabla V\alpha\rho\beta = S\alpha\beta,$$
$$\nabla S\alpha\rho V\beta\rho = -\,S\alpha\beta\rho + 3\beta S\alpha\rho - \rho S\alpha\beta,$$
$$\nabla V\alpha U\rho = (\alpha + \rho^{-1}S\alpha\rho)/T\rho, \qquad \nabla\cdot TV\alpha\rho = UV\alpha\rho\cdot\alpha,$$
$$\nabla TV\rho V\alpha\rho = (V\alpha\rho + \alpha\rho)UV\rho V\alpha\rho,$$
$$\nabla V\alpha\rho/T^3\rho = (\alpha\rho^2 - 3\rho S\alpha\rho)/T^5\rho,$$

$$\nabla\cdot UV\alpha\rho = \frac{\alpha}{TV\alpha\rho}, \qquad \nabla UV\rho V\alpha\rho = \frac{-\,\alpha U\rho}{TV\alpha\rho},$$

$$\nabla(V\alpha\rho)^{-1} = 0, \qquad \nabla\left(\frac{U\rho}{T^2\rho}\right) = 0.$$

EXERCISE

Show that $(V\alpha\beta\cdot\Phi\gamma + V\theta\gamma\cdot\Phi\alpha + V\gamma\alpha\cdot\Phi\beta)/S\alpha\beta\gamma$ is independent of α, β, γ, where Φ is any rational linear function (scalar, vector, or quaternion) of the vector following it. If $\Phi = S\delta(\,) + \Sigma\alpha_1S\beta_1(\,)$ the expression is $\delta + \Sigma\beta_1\alpha_1$.

NOTATION FOR DERIVATIVES OF VECTORS

Directional derivative

$-\,Sa\nabla$, Tait, Joly.

$a\cdot\nabla$, Gibbs, Wilson, Jaumann, Jung.

$\dfrac{d\cdot}{dP}\cdot a$, Burali-Forti, Marcolongo.

Circuital derivative

$Va\nabla$, Tait, Joly.
$a \times \nabla$, Gibbs, Wilson, Jaumann, Jung.

Projection of directional derivative on the direction.

$S \cdot a^{-1}\nabla Sau$, Tait, Joly.

$\frac{\partial \cdot}{\partial a}$, Fischer.

Projection of directional derivative perpendicular to the direction

$V \cdot a^{-1}u'S\nabla' a$, Tait, Joly.

$\frac{\partial \times}{\partial a}$, Fischer.

Gradient of a scalar

∇, Tait, Joly, Gibbs, Wilson, Jaumann, Jung, Carvallo, Bucherer.
grad, Lorentz, Gans, Abraham, Burali-Forti, Marcolongo, Peano, Jaumann, Jung.
$-$ *grad*, Jahnke, Fehr.

$\frac{d}{dr}$, Fischer. [Fischer's multiplication follows Gibbs, d/dr being after the operand, the whole being read from right to left; e.g., Fischer's ∇v is equiv. to $-vS\nabla$.]

Gradient of a vector

∇, Tait, Joly, Gibbs, Wilson, Jaumann, Jung, Carvallo.
grad, Jaumann, Jung.

$\frac{d}{dr}$, Fischer.

7. **Directional Derivative.** One of the most important operators in which ∇ occurs is $-S\alpha\nabla$, which gives the

rate of variation of a function in the direction of the unit vector α. The operation is called directional differentiating.

$$S\alpha\nabla\cdot S\beta\rho = -S\alpha\beta, \qquad S\alpha\nabla\cdot\rho^2 = -2S\alpha\rho,$$
$$S\alpha\nabla T\rho = S\alpha U\rho, \qquad S\alpha\nabla T\rho^{-1} = -S\alpha\rho/T\rho^3 = 1!Y_1 T\rho^{-2},$$
$$S\alpha\nabla TV\alpha\rho = 0, \qquad S\alpha\nabla\cdot U\rho = -\frac{V\rho V\alpha\rho}{T\rho^3}.$$

An iteration of this operator upon $T\rho^{-1}$ gives the series of rational spherical and solid harmonics as follows:

$$-S\alpha\nabla T\rho^{-1} = -S\alpha\rho/T\rho^3 = 1!Y_1 T\rho^{-2},$$
$$S\beta\nabla S\alpha\nabla T\rho^{-1} = (3S\alpha\rho S\beta\rho + T\rho^2 S\alpha\beta)T\rho^{-5} = 2!Y_2 T\rho^{-3},$$
$$S\gamma\nabla S\beta\nabla S\alpha\nabla T\rho^{-1} = -(3.5S\alpha\rho S\beta\rho S\gamma\rho + 3\Sigma S\beta\gamma S\alpha\rho T\rho^2)T\rho^{-7} = 3!Y_3 T\rho^{-4}.$$

For an n axial harmonic we apply n operators, giving

$$Y_n = \Sigma_s(-1)^s(2n-2s)!/[2^{n-s}n!(n-s)!]\Sigma S^{n-2s}\alpha U\rho S^s\alpha_1\alpha_2, \quad 0 \gtreqqless s \gtreqqless n/2.$$

The summation runs over $n - 2s$ factors of the type $S\alpha_1 U\rho S\alpha_2 U\rho\cdots$ and s factors of the type $S\alpha_i\alpha_j S\alpha_k\alpha_l\cdots$, each subscript occurring but once in a given term. The expressions Y are the surface harmonics, and the expressions arising from the differentiation are the solid harmonics of negative order. When multiplied by $T\rho^{2n+1}$ we have corresponding solid harmonics of positive order.

The use of harmonics will be considered later.

8. **Circuital Derivative.** Another important operator is $V\alpha\nabla$ called the *circuital derivative.* It gives the areal density of the circulation, that is to say, if we integrate the function combined with $d\rho$ in any linear way, around an infinitesimal loop, the limit of the ratio of this to the area of the loop is the *circuital derivative,* α being the normal to the area. We give a few of its formulae. We may also

find it from the differential, for if $dQ = \Phi d\rho$, $V\alpha\nabla \cdot Q = V \cdot \alpha\zeta \cdot \Phi\zeta$.

$$V\alpha\nabla \cdot T\rho = V\alpha U\rho, \qquad V\alpha\nabla \cdot T\rho^n = nT\rho^{n-2}V\alpha\rho,$$
$$V\alpha\nabla \cdot U\rho = (3\alpha\rho^2 - \rho S\alpha\rho)/T\rho^3, \qquad V\alpha\nabla \cdot S\beta\rho = V\beta\alpha,$$
$$V\alpha\nabla \cdot V\beta\rho = \alpha\beta + S \cdot \alpha\beta, \qquad V\alpha\nabla \cdot \beta\rho = 2S\alpha\beta,$$
$$V\alpha\nabla \cdot TV\beta\rho = -V \cdot \alpha\beta UV\beta\rho, \qquad V\alpha\nabla \cdot \rho = -2\alpha,$$
$$V\alpha\nabla \cdot (\alpha_1 S\beta_1\rho + \alpha_2 S\beta_2\rho + \alpha_3 S\beta_3\rho) = S\alpha(\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) + V\alpha_1 V\alpha\beta_1 + V\alpha_2 V\alpha\beta_2 + V\alpha_3 V\alpha\beta_3.$$

9. **Solutions of** $\nabla Q = 0$, $\nabla^2 Q = 0$. In a preceding formula we saw that $\nabla(V\alpha\rho)^{-1} = 0$. We can easily find a number of such vectors, for if we apply $S\alpha\nabla$ to any vector of this kind we shall arrive at a new vector of the same kind. The two operators ∇ and $S\alpha\nabla \cdot$ are commutative in their operation. For instance we have

$$d(V\alpha\rho)^{-1} = -(V\alpha\rho)^{-1}V\alpha d\rho \cdot (V\alpha\rho)^{-1};$$

hence

$$\tau = S\beta\nabla \cdot (V\alpha\rho)^{-1} = (V\alpha\rho)^{-1}V\beta\alpha \cdot (V\alpha\rho)^{-1}$$

is a new vector which gives $\nabla\tau = 0$. The series can easily be extended indefinitely. Another series is the one derived from $U\rho/T^2\rho$. This vector is equal to $\rho/T^3\rho$, and its differential is

$$(-\rho^2 d\rho + 3Sd\rho\rho \cdot \rho)/T^5\rho.$$

The new vector for which the gradient vanishes is then

$$(-\alpha\rho^2 + 3S\alpha\rho \cdot \rho)/T^5\rho.$$

The latter case however is easily seen to arise from the vector $\nabla T\rho^{-1}$, and hence is the first step in the process of using ∇ twice, and it is evident that $\nabla^2 T\rho^{-1} = 0$. So also the first case above is the first step in applying ∇^2 to log $TV\alpha\rho \cdot \alpha^{-1}$ so that $\nabla^2(\log TV\alpha\rho \cdot \alpha) = 0$. Functions of ρ that satisfy this partial differential equation are called

harmonic functions. That is, $f(\rho)$ is harmonic if $\nabla^2 f(\rho) = 0$. Indeed if we start with any harmonic scalar function of ρ and apply ∇ we shall have a vector whose gradient vanishes, and it will be the beginning of a series of such vectors produced by applying $S\alpha_1\nabla$, $S\alpha_2\nabla$, $\cdots$ to it. However we may also apply the same operators to the original harmonic function deriving a series of harmonics. From these can be produced a series of vectors of the type in question. $\nabla^2 \cdot F(\rho)$ is called the *concentration* of $F(\rho)$. The concentration vanishes for a harmonic function.

EXERCISES

Show that the following are harmonic functions of ρ:

1. $$T\rho^{-1} \tan^{-1} S\alpha\rho / S\beta\rho,$$

where α and β are perpendicular unit vectors,

2. $$T\rho^{-1} \log \tan \frac{1}{2} \angle \frac{\rho}{\alpha}.$$

3. $$\tan^{-1} S\alpha\rho / S\beta\rho$$

where

$$S\alpha\beta = 0$$

and

$$\alpha^2 = \beta^2 = -1.$$

4. $$\log \tan \frac{1}{2} \angle \frac{\rho}{\alpha}.$$

10. **Harmonics.** We may note that if u, v are two scalar functions of ρ, then

$$\nabla \cdot uv = u\nabla v + v\nabla u$$

and thus

$$\nabla^2 \cdot uv = u\nabla^2 v + v\nabla^2 u + 2S\nabla u\nabla v.$$

Hence the product of two harmonics is not necessarily harmonic, unless the gradient of each is perpendicular to the gradient of the other.

Also if u is harmonic, then

$$\nabla^2 \cdot uv = u\nabla^2 v + 2S\nabla u\nabla v.$$

If u is harmonic and of degree n homogeneously in ρ, then $u/T\rho^{2n+1}$ is a harmonic of degree $-(n+1)$. For

$$\nabla^2(T\rho^{2n+1})^{-1} = \nabla[-(2n+1)\rho T\rho^{-2n-3}]$$
$$= -(2n+1)(2n)T\rho^{-2n-3}$$

and

$$S\nabla u\nabla T\rho^{-2n-1} = -(2n+1)T\rho^{-2n-3}S\nabla u\rho$$
$$= (2n+1)(2n)uT\rho^{-2n-3};$$

hence

$$\nabla^2 \cdot u/T\rho^{2n+1} = 0.$$

In this case u is a solid harmonic of degree n and $uT\rho^{-2n-1}$ is a solid harmonic of degree $-n-1$. Also $uT\rho^{-n}$ *is a corresponding surface harmonic.* The converse is true.

EXAMPLES OF HARMONICS

Degree $n=0$; $\varphi = \tan^{-1}\dfrac{S\alpha\rho}{S\beta\rho}$

$$\text{where } S\alpha\beta = 0,\ \alpha^2 = \beta^2 = -1;$$

$$\psi = \log\cot\tfrac{1}{2}\angle\frac{\rho}{\alpha}\alpha^2 = -1;$$

$$S\cdot\alpha\beta U\rho S\alpha\rho S\beta\rho/V^2\cdot\alpha\beta\rho;$$

$$S\alpha\beta U\rho S(\alpha+\beta)\rho S(\alpha-\beta)\rho/V^2\alpha\beta\rho.$$

The gradients of these as well as the result of any operation $S\gamma\nabla$ are solid harmonics of degree -1, hence multiplying the results by $T\rho[n=1,\ 2n-1=1]$ gives harmonics again of degree 0. These will be, of course, rational harmonics but not integral.

Taking the gradient again or operating by $S\gamma_1\nabla$ any number of times will give harmonics of higher negative degree. Multiplying any one of degree $-n$ by $T\rho^{2n-1}$ will give a solid harmonic of degree $n-1$.

Degree $n=-1$. Any harmonic of degree 0 divided by $T\rho$, for example,

$$1/T\rho, \qquad \varphi/T\rho, \qquad \psi/T\rho, \qquad S\alpha\beta U\rho S\alpha U\rho S\beta\rho/V^2\alpha\beta\rho, \cdots,$$

Degree $n = -2$.

$$S\alpha U\rho/\rho^2, \qquad \varphi S\alpha\beta U\rho/\rho^2, \qquad \psi S\alpha\beta U\rho/\rho^2 + \rho^{-2} \cdots.$$

Degree $n = 1$.

$$S\alpha\rho, \qquad \varphi S\alpha\beta\rho, \qquad \psi S\alpha\beta\rho + T\rho \cdots.$$

Other degrees may easily be found.

11. **Rational Integral Harmonics.** The most interesting harmonics from the point of view of application are the rational integral harmonics. For a given degree n there are $2n + 1$ independent rational integral harmonics. If these are divided by $T\rho^n$ we have the spherical harmonics of order n. When these are set equal to a constant the level surfaces will be cones and the intersections of these with a unit sphere give the lines of level of the spherical harmonics of the given order. A list of these follow for certain orders. Drawings are found in Maxwell's *Electricity and Magnetism.*

Rational integral harmonics, Degree 1. $S\alpha\rho$, $S\beta\rho$, $S\gamma\rho$, α, β, γ a trirectangular unit system.

Degree 2. $S\alpha\rho S\beta\rho$, $S\beta\rho S\gamma\rho$, $S\gamma\rho S\alpha\rho$, $3S^2\alpha\rho + \rho^2$, $S^2\alpha\rho - S^2\beta\rho$.

These correspond to the operators $T\rho^5[S^2\gamma\nabla$, $S\gamma\nabla S\alpha\nabla$, $S\gamma\nabla S\beta\nabla$, $S(\alpha + \beta)\nabla S(\alpha - \beta)\nabla$, $S\alpha\nabla S\beta\nabla]$ on $T\rho^{-1}$.

Degree 3. Representing $S\alpha\rho$ by $-x$, $S\beta\rho$ by $-y$, $S\gamma\rho$ by $-z$, $S\alpha\nabla$ by $-D_x$, $S\beta\nabla$ by $-D_y$, $S\gamma\nabla$ by $-D_z$ we have

$$2z^3 - 3x^2z - 3y^2z, \qquad 4z^2x - x^3 - y^2x, \qquad 4z^2y - x^2y - y^3,$$
$$x^2z - y^2z, \qquad xyz, \qquad x^3 - 3xy^2, \qquad 3x^2y - y^3$$

corresponding to

$$D_{zzz}{}^3, \quad D_{zzx}{}^3, \quad D_{zzy}{}^3, \quad D_{xxz}{}^3, \quad -D_{xyz}{}^3, \quad D_{xxx}{}^3, \quad -3D_{xyy}{}^3,$$
$$D_{yyy}{}^3, \qquad -3D_{xxy}{}^3.$$

Degree 4.

$$3x^4 + 3y^4 + 8z^4 + 6x^2y^2 - 24x^2z^2 - 24y^2z^2,$$
$$xz(4z^2 - 3x^2 - 3y^2), \qquad yz(4z^2 - 3x^2 - 3y^2)$$
$$(x^2 - y^2)(6z^2 - x^2 - y^2), \qquad xy(6z^2 - x^2 - y^2),$$
$$xz(x^2 - 3y^2), \qquad yz(3x^2 - y^2), \qquad x^4 + y^4 - 6x^2y^2,$$
$$xy(x^2 - y^2)$$
$$D_{zzzz}{}^4, \qquad D_{zzzx}{}^4, \qquad D_{zzzy}{}^4$$
$$D_{zzxx}{}^4 - D_{zzyy}{}^4, \qquad D_{zzxy}{}^4, \qquad D_{xxxz}{}^4 - 3D_{xyyz}{}^4,$$
$$D_{yyyz}{}^4 - 3D_{xxyz}{}^4, \qquad D_{xxxx}{}^4 + D_{yyyy}{}^4 - 6D_{xxyy}{}^4,$$
$$D_{xxxy}{}^4 - D_{xyyy}{}^4.$$

The curves of the intersections of these cones with the unit sphere are inside of zero-lines as follows:

Degree 1. Equator, standard meridian, longitude 90°.

Degree 2. Latitudes $\pm \sin^{-1} \frac{1}{3}\sqrt{3}$, equator and standard meridian, equator and longitude 90°, longitude $\pm$ 45°. Standard meridian and 90°.

Degree 3. Latitudes 0°, $\pm \sin^{-1} \sqrt{0.6}$, latitudes $\pm \sin^{-1} \sqrt{0.2}$ and standard meridian, latitudes $\pm \sin^{-1} \sqrt{0.2}$ and 90° longitude, equator, longitude $\pm$ 45°, equator, longitudes 0°, 90°. Longitudes $\pm$ 30°, 90°, longitudes $\pm$ 60°, 0°.

12. **Variable System of Trirectangular Unit Vectors.** We will consider next a field which contains at every point a system of three lines which are mutually perpendicular. That is, the lines in one direction are given by α, say, at the same point another set by β, and the third set by γ. Each is a given function of ρ, subject to the conditions

$$\alpha\beta = \gamma, \qquad \beta\gamma = \alpha, \qquad \gamma\alpha = \beta, \qquad \alpha^2 = \beta^2 = \gamma^2 = -1.$$

For example, in the ordinary congruence, α being the unit tangent at any point of one line of the congruence, then the normal and the binormal are determined and would be β and γ. However β and γ may be other perpendicular

lines in the plane normal to α. If we follow the vector line for β after we leave the point we shall get a determinate curve, provided we consider α to be its normal. We may however draw any surface through the point which has α for its normal and then on the surface draw any curve through the point. All such curves can serve as β curves but α might not be their principal normal. It can happen therefore that the β curves and the γ curves may start out from the point on different surfaces. However α, β, and γ are definite functions of the position of the point P, with the condition that they are unit vectors and mutually perpendicular.

If we go to a new position infinitesimally close, α becomes $\alpha + d\alpha$, β becomes $\beta + d\beta$, and γ becomes $\gamma + d\gamma$. The new vectors are unit vectors and mutually perpendicular, hence we have at once

$$\begin{gathered} S\cdot\alpha d\alpha = S\cdot\beta d\beta = S\cdot\gamma d\gamma = 0, \qquad S\alpha d\beta = -\,S\beta d\alpha, \\ S\beta d\gamma = -\,S\gamma d\beta, \qquad S\gamma d\alpha = -\,S\alpha d\gamma. \end{gathered} \tag{1}$$

These equations are used frequently in making reductions. We have likewise since $\alpha^2 = -1$,

$$\begin{gathered} \nabla\alpha\cdot\alpha = -\,\nabla'\alpha\alpha', \qquad \nabla\beta\cdot\beta = -\,\nabla'\beta\beta', \\ \nabla\gamma\cdot\gamma = -\,\nabla'\gamma\gamma', \end{gathered} \tag{2}$$

where the accent on the ∇ indicates that it operates only on the accented symbols following. Similarly we have

$$\nabla\alpha\cdot\beta + \nabla\beta\cdot\alpha = -\,\nabla'\alpha\beta' - \nabla'\beta\alpha', \qquad \text{etc.} \tag{3}$$

We notice also that

$$\begin{gathered} S\cdot\alpha(S()\nabla)\alpha = 0, \\ S\cdot\alpha(S()\nabla)\beta = -\,S\cdot\beta(S()\nabla)\alpha, \qquad \text{etc.} \end{gathered} \tag{4}$$

We now operate on the equation $\gamma = \alpha\beta$ with ∇, and

remember that for any two vectors $\lambda\mu$ we have $\lambda\mu = -\mu\lambda + 2S\lambda\mu$, whence

$$\nabla\gamma = \nabla\alpha\cdot\beta + \nabla'\alpha\beta' = \nabla\alpha\cdot\beta - \nabla\beta\cdot\alpha + 2\nabla'S\alpha\beta'. \quad (5)$$

The corresponding equations for the other two vectors are found by changing the letters cyclically.

Multiply every term into γ and we have

$$\nabla\gamma\cdot\gamma = \nabla\alpha\cdot\alpha + \nabla\beta\cdot\beta + 2\nabla'S\alpha\beta'\cdot\gamma. \quad (6)$$

If now we take the scalar of both sides we have

$$S\gamma\nabla\gamma = S\alpha\nabla\alpha + S\beta\nabla\beta + 2S\gamma\nabla'S\alpha\beta'. \quad (7)$$

We set now

$$2p = + S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma \quad (8)$$

and the equation (7) gives, with the similar equations deduced by cyclic interchange of the letters,

$$\begin{aligned} S\gamma\nabla'S\alpha\beta' &= -S\gamma\nabla'S\alpha'\beta = -p + S\gamma\nabla\gamma, \\ S\alpha\nabla'S\beta\gamma' &= -S\alpha\nabla'S\beta'\gamma = -p + S\alpha\nabla\alpha, \\ S\beta\nabla'S\gamma\alpha' &= -S\beta\nabla'S\gamma'\alpha = -p + S\beta\nabla\beta, \\ -S\cdot\gamma[-S\alpha\nabla\cdot\gamma] &= S\alpha\nabla\cdot S\gamma\gamma' = \tfrac{1}{2}S\alpha\nabla\cdot\gamma^2 = 0, \\ -S\cdot\alpha[-S\alpha\nabla\cdot\gamma] &= -S\alpha\nabla\cdot S\alpha'\gamma \\ &= S\gamma(-S\alpha\nabla\cdot\alpha) = S\gamma(u\beta + v\gamma) = -v. \end{aligned} \quad (9)$$

That is to say, the rate of change of γ, if the point is moved along α, is $\beta(S\alpha\nabla\alpha - p)$. Likewise

$$d\beta/ds = -\gamma(-p + S\alpha\nabla\alpha) - v\alpha.$$

The trihedral therefore rotates about α with the rate $(p - S\alpha\nabla\alpha)$ as its vertex moves along α. Now we let

$$t_\alpha = +p - S\alpha\nabla\alpha. \quad (10)$$

We may also write at once, similarly,

$$t_\beta = +p - S\beta\nabla\beta, \qquad t_\gamma = +p - S\gamma\nabla\gamma, \quad (10)$$

from which we derive

$$t_\alpha + t_\beta + t_\gamma = +p. \quad (11)$$

It is also evident that

$$t_\alpha + t_\beta = S\gamma\nabla\gamma, \qquad t_\beta + t_\gamma = S\alpha\nabla\alpha, t_\gamma + t_\alpha = S\beta\nabla\beta. \quad (12)$$

The expressions on the left hold good for any two perpendicular unit vectors in the plane normal to the vector on the right, and hence if we divide each by 2 and call the result the mean rotatory deviation for the trajectories of the vector on the right, we have

$$\tfrac{1}{2}S\alpha\nabla\alpha = \text{mean rotatory deviation for } \alpha.$$

Again the negative rotation for the β trajectory gives what we have called previously the rotatory deviation of α along β. Hence, as a similar statement holds for γ, the mean rotatory deviation is one half the sum of the rotatory deviations. Hence $\tfrac{1}{2}S\alpha\nabla\alpha$ is the negative rate of rotation of the section of a tube of infinitesimal size, whose central trajectory is α, about α, as the point moves along α. Or we may go back to (9) and see that

$$\begin{aligned} S\alpha\nabla\alpha = (+\,p - S\beta\nabla\beta) + (+\,p - S\gamma\nabla\gamma) \\ = -\,S\beta\nabla' S\gamma\alpha' + S\gamma\nabla' S\beta\alpha', \end{aligned}$$

which gives the rotatory deviations directly.

The scalar of (5) and the like equations are

$$\begin{gathered} S\nabla\alpha = S\gamma\nabla\beta - S\beta\nabla\gamma, \quad S\nabla\beta = S\alpha\nabla\gamma - S\gamma\nabla\alpha, \\ S\nabla\gamma = S\beta\nabla\alpha - S\alpha\nabla\beta, \end{gathered} \quad (13)$$

We multiply next (5) by α and take the scalar, giving

$$\begin{aligned} S\gamma\nabla\alpha &= -\,S\alpha\nabla' S\beta\alpha' = S\alpha\nabla' S\alpha\beta', \\ S\beta\nabla\alpha &= -\,S\alpha\nabla' S\alpha\gamma' = S\alpha\nabla' S\gamma\alpha', \\ S\alpha\nabla\beta &= -\,S\beta\nabla' S\gamma\beta' = S\beta\nabla' S\beta\gamma', \\ S\gamma\nabla\beta &= -\,S\beta\nabla' S\beta\alpha' = S\beta\nabla' S\alpha\beta', \\ S\beta\nabla\gamma &= -\,S\gamma\nabla' S\alpha\gamma' = S\gamma\nabla' S\gamma\alpha', \\ S\alpha\nabla\gamma &= -\,S\gamma\nabla' S\gamma\beta' = S\gamma\nabla' S\beta\gamma'. \end{aligned} \quad (14)$$

We can therefore write

$$S\nabla\alpha = -S\beta\nabla' S\beta\alpha' - S\gamma\nabla' S\gamma\alpha',$$

that is $S\nabla\alpha$ equals the negative sum of the projection of the rate of change of α along β on β, and the rate of change of α along γ on γ. But these are the divergent deviations of α and hence $-S\nabla\alpha$ is the geometric divergence of the section. It gives the rate of the expansion of the area of the cross-section of the tube around α. We may write the corresponding equations of β and γ.

Again we have

$$\begin{aligned} V\nabla\alpha &= -\alpha S\alpha\nabla\alpha - \beta S\beta\nabla\alpha - \gamma S\gamma\nabla\alpha \\ &= \alpha(t_\alpha - p) - \beta S\gamma(S\alpha\nabla\cdot\alpha) + \gamma S\beta(S\alpha\nabla\cdot\alpha) \\ &= \alpha(t_\alpha - p) - V\alpha(S\alpha\nabla\cdot\alpha). \end{aligned}$$

Now from the Frenet formulae

$$-S\alpha\nabla\cdot\alpha = c_\alpha\nu,$$

where c_α is the curvature of the trajectory and ν is the principal normal. Hence

$$V\nabla\alpha = \alpha(t_\alpha - p) + c_\alpha\mu, \tag{15}$$

where μ is the binormal of the trajectory. We find therefore that $V\nabla\alpha$ consists of the sum of two vectors of which one is twice the rate of rotation of the section or an elementary cube about α, measured along α, and the other is twice the rate of rotation of the elementary cube about the binormal measured along the binormal.* But we will see

* This should not be confused with the rotation of a rigid area moving along a curve. The infinitesimal area changes its shape since each point of it has the same velocity. As a deformable area it rotates (i.e. the invariant line of the deformation) with half the curvature as its rate. The student should picture a circle as becoming an ellipse, which ellipse also rotates about its center.

later that this sum is the vector which represents twice the rate of rotation of the cube and the axis as it moves along the trajectory of α. Hence this is what we have called the geometric curl.

We may now consider any vector σ defining a vector field not usually a unit vector. Since $\sigma = T\sigma U\sigma$, we have

$$S\nabla\sigma = SU\sigma\nabla T\sigma + T\sigma S\nabla U\sigma.$$

The last term is the geometric convergence multiplied by the length of σ, that is, it is the convergence of a section at the end of σ. The first term is the negative rate of change of $T\sigma$ along σ. The two together give therefore the rate of decrease of an infinitesimal volume cut off from the vector tube, as it moves along the tube. In the language of physics, this is the *convergence* of σ. Similarly we have

$$V\nabla\sigma = V\nabla T\sigma U\sigma + T\sigma V\nabla U\sigma.$$

The last term is the double rate of rotation of an elementary cube at the end of σ, while the first term is a rotation about that part of the gradient of $T\sigma$ which is perpendicular to $U\sigma$. It is, indeed, for a small elementary cube a shear of one of the faces perpendicular to $U\sigma$, which gives, as we have seen, twice the rate of rotation corresponding. Consequently $V\nabla\sigma$ is twice the vector rotation of the elementary cube.

EXAMPLES

(1) Show that

$$\alpha S\nabla\alpha + \beta S\nabla\beta + \gamma S\nabla\gamma$$
$$= -V\alpha V\nabla\alpha - V\beta V\nabla\beta - V\gamma V\nabla\gamma.$$

(2) Show that if $c_1(\alpha)$ is zero $V\alpha V\nabla\alpha = 0$. This is the condition that the lines of the congruence be straight. It is necessary and sufficient.

(3) Let $V\nabla\alpha = \xi$, $-S\alpha\nabla\alpha = x$, then $T\xi = \sqrt{[c^2 + x^2]}$.

$\xi_1 = -S\alpha\nabla\cdot\xi = x_1\alpha + c_1p\beta + c_2\gamma$, where the subscript 1 means differentiation as to s, that is, along a line of the congruence.

$$-S\beta\xi_1 = c_1p; \qquad a_1 = c_1^{-1}S\beta\xi_1 + x,$$

or

$$p = -\frac{1}{c_1}S\beta\xi_1.$$

This gives the torsion in terms of the curl of α and its derivative.

(4) If the curves of the congruence are normals to a set of surfaces, then

$$\alpha = U\nabla u \quad \text{and} \quad \nabla\alpha = \nabla^2 u/T\nabla u - \nabla(1/T\nabla u)\cdot\nabla u.$$

Hence we have at once $S\alpha\nabla\alpha = 0 = x$. This condition is necessary and sufficient.

(5) If also $V\alpha V\nabla\alpha = 0$, we have a Kummer normal system of straight rays. In this case by adding the two conditions, $\alpha V\nabla\alpha = 0$, that is, $V\nabla\alpha = 0$. This condition is also necessary and sufficient.

(6) If the curves are plane, $a_1 = 0$ or $S\alpha\nabla\alpha = S\beta\nabla\beta + S\gamma\nabla\gamma$ or $S\beta\xi_1 = -xc_1$, which is necessary and sufficient.

(7) If further they are normal to a set of surfaces $S\beta\nabla\beta + S\gamma\nabla\gamma = 0 = S\beta\xi_1$. The converse holds.

(8) If c_1 is constant, $S\gamma\xi_1 = 0$ and conversely.

If also plane, and therefore circles, $S\beta\xi_1 = 0$ or $\xi_1 = x_1\alpha + c_1x\beta$. This is necessary and sufficient.

For a normal system of circles we have also

$$V\nabla\alpha = \text{const} = c_1\gamma.$$

(9) For twisted curves of constant curvature $\xi_1 = -c_1a_1\beta$.

Notations

Vortex of a vector

$V\nabla u$, Tait, Joly, Heaviside, Föppl, Ferraris.

$\nabla \times u$, Gibbs, Wilson, Jaumann, Jung.

curl u, Maxwell, Jahnke, Fehr, Gibbs, Wilson, Heaviside, Föppl, Ferraris. *Quirl* also appears.

$[\nabla u]$, Bucherer.

rot u, Jaumann, Jung, Lorentz, Abraham, Gans, Bucherer.

$\frac{1}{2}$ *rot* u, Burali-Forti, Marcolongo.

$\dfrac{d \times u}{dr}$, Fischer.

Vort u, Voigt.

(Notations corresponding to $V\nabla u$ are also in use by some that use *curl* or *rot*.)

Divergence of a vector

$-S\nabla u$, Tait, Joly. $S\nabla u$ is the "convergence" of Maxwell.

$\nabla \cdot u$, Gibbs, Wilson, Jaumann, Jung.

div u, Jahnke, Fehr, Gibbs, Wilson, Jaumann, Jung, Lorentz, Bucherer, Gans, Abraham, Heaviside, Föppl, Ferraris, Burali-Forti, Marcolongo.

∇u, Lorentz, Abraham, Gans, Bucherer.

$\dfrac{d \cdot u}{dr}$, Fischer.

Derivative dyad of a vector

$-S()\nabla \cdot u$, Tait, Joly.

$\cdot \nabla u$, Gibbs, Wilson.

$\cdot \nabla ; u$, Jaumann, Jung.

$\dfrac{du}{dP}$, Burali-Forti, Marcolongo.

$\dfrac{du}{dr}$, Fischer.

$D_u \cdot$, Shaw.

Conjugate derivative dyad of a vector

$-\nabla Su()$, Tait, Joly.

$\nabla u\cdot$, Gibbs, Wilson.

$\nabla ; u\cdot$, Jaumann, Jung.

$K_1()$, Burali-Forti, Marcolongo.

$\frac{du}{dr_c}$, Fischer.

$\breve{D}_u\cdot$, Shaw.

Planar derivative dyad of a vector

$V\nabla Vu()$, Tait, Joly.

$\nabla\times(u\times())$, Gibbs, Wilson.

$\nabla \overset{\times}{,} u$, Jaumann, Jung.

$CK\frac{du}{dP}$, Burali-Forti, Marcolongo.

$-x(D_u)$, Shaw.

Dispersion. Concentration

$-\nabla^2$, Tait, Joly. ∇^2 is the "concentration" of Maxwell.

∇^2, Lorentz, Abraham, Gans, Bucherer.

$\nabla\cdot\nabla$, Gibbs, Wilson, Jaumann, Jung.

div grad, Fehr, Burali-Forti, Marcolongo.

$-$ *div grad*, Jahnke.

$\left.\begin{array}{l}\Delta_2, \text{ for scalar operands,}\\ \Delta_2', \text{ for vector operands,}\end{array}\right\}$ Burali-Forti, Marcolongo.

$\frac{d\cdot d}{dr^2}$, Fischer.

Dyad of the gradient. Gradient of the divergence

$-\nabla S\nabla$, Tait, Joly.

$\nabla\nabla\cdot$, Gibbs, Wilson.

$\nabla ; \nabla$, Jaumann, Jung.

grad div, Buroli-Forti, Marcolongo.

Planar dyad of the gradient. Vortex of the vortex

$V\nabla V\nabla()$, Tait, Joly.

$\nabla \overset{\times}{,} \nabla$, Jaumann, Jung.

*rot*², Lorentz, Bucheoer, Gons, Abraham.

*curl*², Heaviside, Föppl, Ferraris.

rot rot, Burali-Forti, Marcolongo.

13. **Vector Potential, Solenoidal Field.** If $\xi = V\nabla\sigma$, then we say that σ is a *vector potential of* ξ. Obviously

$$S\nabla\xi = SV^2\sigma = 0.$$

The vector potential is not unique, since to it may be added any vector of vanishing curl. When the convergence of a vector vanishes for all values of the vector in a given region we call the vector *solenoidal*. If the curl vanishes then the vector is *lamellar*.

We have an example of lamellar fields in the vector field which is determined by the gradient of any scalar function, for $V\nabla\nabla u = 0$.

In case the field of a unit vector is solenoidal we see from the considerations of § 12 that the first and second divergent deviations of any one of its vector lines are opposite. If then we draw a small circuit in the normal plane of the vector line at P and at the end of $d\rho$ a second circuit in the normal plane at $\rho + d\rho$, and if we project this second circuit back upon the first normal plane, then the second will overlie the first in such a way that if from P a radius vector sweeps out this circuit then for every position in which the radius vector must be extended to reach the second circuit there is a corresponding position at right angles to it in which it must be shortened by an equal amount. It follows that the limit of the ratio of the areas of the two circuits is unity. Hence if such a vector tube is followed throughout the field it will have a constant cross-

section. In the general case it is also clear that $S\nabla\sigma$ gives the contraction of the area of the tube.

When σ is not a unit vector then we see likewise that $S\nabla\sigma$ by § 12 has a value which is the product of the contraction in area by the $T\sigma$ + the contraction of $T\sigma$ multiplied by the area of the initial circuit. Hence $S\nabla\sigma$ represents the volume contraction of the tube of σ for length $T\sigma$ per unit area of cross-section. When the field is solenoidal it follows that if $T\sigma$ is decreasing the tubes are widening and conversely.

For instance, $S\nabla U\rho = -2/T\rho$ signifies that per unit length along ρ the area of a circuit which is normal to ρ is increasing in the ratio $2/T\rho$, that is, the flux of $U\rho$ is increasing at the rate of $2/T\rho$ along ρ. Also $S\cdot\nabla\rho = -3$ indicates that an infinitesimal volume taken out of the field of ρ is increasing in the ratio 3. Of this the increase 2 is due to the widening of the tubes, as just stated, the increase 1 is due to the rate at which the intensity of the field is increasing. If the field is a velocity field, the rate of increase of volume of an infinitesimal mass is 3 times per second.

It is evident now if we multiply $S\nabla\sigma$ by a differential volume dv that we have an expression for the differential flux into the volume. If σ is the velocity of a moving mass of air, say unit mass, then $S\nabla\sigma$ is the rate of compression of this moving mass, and $S\nabla\sigma dv$ is the compression per unit time of this mass, and $\iiint S\nabla\sigma dv$ is the increase in mass per unit time of matter at initial density or compression per unit time of a given finite mass which occupies initially the moving volume furnishing the boundary of the integral.

If τ is the specific momentum or velocity of unit volume times the density, then $S\nabla\tau$ is the condensation rate or

rate of increase of the density at a given fixed point, and $S\nabla\tau dv$ is the increase in mass in dv per unit time. Hence $\iiint S\nabla\tau dv$ is the increase in mass per unit time in a given fixed space.

Since

$$\sigma = \frac{1}{c}\tau$$

where c is density at a point,

$$S\nabla\sigma = -\frac{1}{c^2}S\nabla c\tau + \frac{1}{c}S\nabla\tau$$

$$= -S\sigma\nabla\cdot\log c + \frac{\partial \log c}{\partial t} = \frac{d \log c}{dt}$$

= total relative rate of change of density due to velocity and to time,

= relative rate of change of density at a moving point.

$S\nabla\sigma\cdot dv =$ increase in mass of a moving dv divided by the original density.

$\iiint S\nabla\sigma\cdot dv =$ increase in mass in a moving volume per unit of time divided by original density,

= decrease in volume of an original mass.

For an incompressible fluid $S\nabla\sigma = 0$ or σ is solenoidal, and for a homogeneous fluid $S\nabla\tau = 0$ or τ is solenoidal. In water of differing salinity $S\nabla\sigma = 0$, $S\nabla\tau \neq 0$. We have a case of constant τ in a column of air. If we take a tube of cross-section 1 square meter rising from the ocean to the cirrus clouds, we may suppose that one ton of air enters at the bottom, so that one ton leaves at the top, but the volume at the bottom is 1000 cubic meters and at the top 3000 cubic meters. Hence the volume outflow at the top is 2000 cubic meters. In the hydrosphere σ and τ

are solenoidal, in the atmosphere τ is solenoidal. We measure σ in m^3/sec and τ in tons/m^2 sec. At every stationary boundary σ and τ are tangential, and at a surface of discontinuity of mass, the normal component of the velocity must be the same on each side of the surface, as for example, in a mass of moving mercury and water.

It is evident that if a vector is solenoidal, and if we know by observation or otherwise the total divergent deviations of a vector of length $T\sigma$, then the sum of these will furnish us the negative rate of change of $T\sigma$ along σ. Thus, if we can observe the outward deviations of τ in the case of an air column, we can calculate the rate of change of $T\tau$ vertically. If we can observe the outward deviations of a tube of water in the ocean we can calculate the decrease in forward velocity.

EXERCISES.

1. An infinite cylinder of 20 cm. radius of insulating material of permittivity 2 [farad/cm.], is uniformly charged with $1/20\pi$ electrostatic units per cubic cm. Find the value of the intensity **E** inside the rod, and also outside, its convergence, curl, and if there is a potential for the field, find it.

2. A conductor of radius 20 cm. carries one absolute unit of current per square centimeter of section. Find the magnetic intensity **H** inside and outside the wire and determine its convergence, curl, and potential.

14. **Curl.** We now turn our attention to another meaning of the curl of a vector. We can write the general formula for the curl

$$V\nabla\sigma = -\sigma SU\sigma\nabla U\sigma - \beta S\gamma\nabla T\sigma + \gamma(cT\sigma + S\beta\nabla T\sigma)$$

Let $U\sigma = \sigma'$. These terms we will interpret, one by one. It was shown that the first term is σ multiplied by the sum of the rotational deviations of σ'. But if we consider a small rectangle of sides $\eta dt = d_1\rho$ and $\tau du = d_2\rho$, then the corresponding actual deviations are

$$Sd_1\rho d_2\sigma' \quad \text{and} \quad -Sd_2\rho d_1\sigma'$$

and the sum becomes

$$Sd_1\rho d_2\sigma' - Sd_2\rho d_1\sigma'.$$

But $d_2\sigma'$ is the difference between the values of σ' at the origin and the end of $d_2\rho$, and to terms of first order is the difference of the average values of σ' along the two sides $d_1\rho$ and $d_1\rho + d_2\rho - d_1\rho$. Likewise $d_1\sigma$ is the difference between the average values of σ' along the side $d_2\rho$ and its opposite. Hence if we consider $Sd\rho\sigma'$ for a path consisting of the perimeter of the rectangle, the expression above is the value of this $Sd\rho\sigma'$ for the entire path, that is, is the circulation of σ' around the rectangle. Hence the coefficient

$$-\,SU\sigma\nabla U\sigma$$

is the limit of the quotient of the circulation around $d_1\rho\; d_2\rho$ divided by $dtdu$ or the area of the rectangle.

If we divide any finite area in the normal plane of σ into elementary rectangles, the sum of the circulations of the elements will be the circulation around the boundary, and we thus have the integral theorem

$$\oint Sd\rho\sigma = \int\int Sd_1\rho d_2\rho V\nabla\sigma$$

when $Vd_1\rho d_2\rho$ is parallel to $V\nabla\sigma$. The restriction, we shall see, may be removed as the theorem is always true.

The component of $V\nabla\sigma$ along σ is then

$$-\,U\sigma \operatorname{Lim} \oint Sd\rho\sigma/\text{area of loop}$$

as the area decreases and the plane of the loop is normal to σ.

Consider next the term $-\,\beta S\gamma\nabla T\sigma$. It is easy to reduce to this form the expression

$$[-\,S\sigma'(S\gamma\nabla)\sigma + S\gamma(S\sigma'\nabla)\sigma][-\,\beta].$$

But this is the circulation about a small rectangle in the

plane normal to β. Hence the component of $V\nabla\sigma$ in the direction β is

$$-\beta \text{ Lim } \oint Sd\rho\sigma/\text{area of loop in plane normal to } \beta.$$

Likewise the other term reduces to a similar form and the component of $V\nabla\sigma$ in the direction γ is

$$-\gamma \text{ Lim } \oint Sd\rho\sigma/\text{area of loop in plane normal to } \gamma.$$

It follows if α is any unit vector that the component of $V\nabla\sigma$ along α is

$$-\alpha \text{ Lim } \oint Sd\rho\sigma/\text{area of loop in plane normal to } \alpha$$

as the loop decreases. The direction of $UV\nabla\sigma$ is then that direction in which the limit in question is a maximum, and in such case $TV\nabla\sigma$ is the value of the limit of the circulation divided by the area. That is, $TV\nabla\sigma$ is the maximum circulation per square centimeter.

Another interpretation of $V\nabla\sigma$ is found as follows: Let us suppose that we have a volume of given form and that σ is a velocity such that each point of the volume has an independent velocity given by σ. Then the moving volume will in general change its shape. The point which is originally at ρ will be found at the new point $\rho + \sigma(\rho)dt$. A point near ρ, say $\rho + d\rho$, will be found at $\rho + d\rho + \sigma(\rho + d\rho)dt$, and the line originally from ρ to $\rho + d\rho$ has become instead of $d\rho$,

$$d\rho + dt[\sigma(\rho + d\rho) - \sigma(\rho)] = d\rho - Sd\rho\nabla\cdot\sigma dt.$$

But this can be written

$$d\rho' = d\rho - [\tfrac{1}{2}V\cdot\nabla' d\rho\sigma' + \tfrac{1}{2}d\rho S\nabla\sigma - \tfrac{1}{2}V(V\nabla\sigma)d\rho]dt.$$

This means, however, that we can find three perpendicular axes in the volume in question such that the effect of the

motion is to move the points of the volume parallel to these directions and to subject them to the effect of the term

$$d\rho + \tfrac{1}{2}V(V\nabla\sigma)d\rho\ dt.$$

Now if we consider an infinitesimal rotation about the vector ϵ its effect is given by the form (du being half of the instantaneous angle)

$$(1 + \epsilon du)\rho(1 - \epsilon du) = \rho + 2V\epsilon\rho du;$$

hence the vector joining ρ and $\rho + d\rho$ will become the vector joining $\rho + 2V\epsilon\rho du$ and $\rho + 2V\epsilon\rho du + d\rho + 2V\epsilon d\rho du$, that is, $d\rho$ becomes $d\rho + 2V\epsilon d\rho\ du$. We find therefore that the form above means a rotation about the vector $UV\nabla\sigma$ of amount $\frac{1}{2}TV\nabla\sigma dt$, or in other words $V\nabla\sigma$, when σ is a velocity, gives in its unit part the instantaneous axis of rotation of any infinitesimal volume moving under this law of velocity, and its tensor is twice the angular velocity. For this reason the curl of σ is often called the *rotation*. When $V\nabla\sigma = 0$, σ has the form $\sigma = \nabla u$, and u is called a velocity potential. If σ is not a velocity, we still call u a potential for σ.

EXERCISES.

1. If a mass of water is rotated about a vertical axis at the rate of two revolutions per second, find the stationary velocity. What are the convergence and the curl of the velocity? Is there a velocity potential?
2. If a viscous fluid is flowing over a horizontal plane from a central axis in such way that the velocity, which is radial, varies as the height above the plane, study the velocity.
3. Consider a part of the waterspout problem on page 50.

15. **Vortices.** Since $V\nabla\sigma$ is a vector it has its vector lines, and if we start at any given point and trace the vector line of $V\nabla\sigma$ such line is called a *vortex line*. The field of $V\nabla\sigma$ is called a vortex field. If a vector is lamellar the vector and the field are sometimes called *irrotational*. The

equation of the vortex lines is

$$Vd\rho V\nabla\sigma = 0 = Sd\rho\nabla\cdot\sigma - \nabla' Sd\rho\sigma' = -d\sigma - \nabla' Sd\rho\sigma'.$$

The rate of change of σ then along one of its vortex lines is $-\nabla' S\alpha\sigma'$. Since $S\nabla V\nabla\sigma = 0$, the curl of σ is always solenoidal, that is, an elementary volume taken along the vortex lines has no convergence but merely rotates.

The curl of the curl is $V\nabla V\nabla\sigma = \nabla^2\sigma - \nabla S\nabla\sigma$ and thus if σ is harmonic the curl of the curl is the negative gradient of the convergence, and if the vector is solenoidal, the curl of the curl is the concentration $\nabla^2\sigma$.

EXERCISES

1. If $S\alpha\sigma = 0 = S\alpha\nabla\cdot\sigma$, and if we set $\sigma = V\cdot\alpha\tau$, and determine X so that $\nabla X = \tau$, then $X\alpha$ is a vector potential of the vector σ.

2. Determine the vector lines in the preceding problem for σ. Also show that the derivative of X in any direction perpendicular to α is equal to the component of σ perpendicular to both. What is $\nabla^2 X$?

3. If $\sigma = w\gamma$ and $-S\gamma\nabla\cdot w = 0$, then either $X\alpha$ or $Y\beta$ will be vector potentials of σ where $\beta\gamma = \alpha$ and all are unit vectors and $S\gamma\nabla\cdot X = 0 = S\gamma\nabla\cdot Y$.

4. If the lines of σ are circles whose planes are perpendicular to γ and centers are on $\rho = t\gamma$, and $T\sigma = f(TV\gamma\rho)$, then any vector parallel to γ whose tensor is $F(TV\gamma\rho)$, where $-f = dF/dTV\gamma\rho$ is a vector potential of σ. Is σ solenoidal?

5. If the lines of σ are straight lines perpendicular to γ and radiating from $\rho = t\gamma$ and $T\sigma = f(TV\gamma\rho)$, then what is the condition that σ be solenoidal? If $T\sigma = f(\tan^{-1} TV\gamma\rho/S\gamma\rho)$ σ cannot be solenoidal.

6. If $\sigma = f(S\alpha\rho, S\beta\rho)\cdot V\gamma\rho\cdot\gamma$, then what is $V\nabla\sigma$? Show that if f is a function of $\tan^{-1} S\alpha\rho/S\beta\rho$, that $S\gamma\rho\nabla f$ is a function of the same angle, but if f is a function of $TV\gamma\rho$, $S\gamma\rho\nabla\cdot f = 0$ and no vector of the form $\sigma = f(TV\gamma\rho)V\gamma\rho\cdot\gamma$ can be a potential of $\gamma TV\gamma\rho$. If $\mu = S\alpha\rho/S\beta\rho$, then $f(\mu) = -\int\varphi(\mu)d\mu/(\mu^2+1)$.

7. What are the lines of $\sigma = f(S\alpha\rho, S\beta\rho)V\gamma\rho$ and what is the curl? If f is a function of $TV\gamma\rho$, so is the curl, and if

$$F(TV\gamma\rho) = (TV\gamma\rho)^{-2}\int TV\gamma\rho\varphi TV\gamma\rho dTV\gamma\rho$$

then $F\cdot TV\gamma\rho$ is a vector potential of the solenoidal vector $\gamma\varphi T(V\gamma\rho)$. If f is a function of μ the curl is a function of μ, and $\frac{1}{2}f(\mu)V\gamma\rho$ is a vector potential of $\gamma f(\mu)$.

8. If σ is solenoidal and harmonic the curl of its curl is zero. If its

lines are plane and it has the same tensor at all points in a line perpendicular to the plane, then it is perpendicular to its curl.

9. The vector $\sigma = f\cdot U\rho$, where f is any scalar function of ρ, is not necessarily irrotational, but $S\sigma\nabla\sigma = 0$.

10. If a vector is a function of the two scalars $S\lambda\rho$, $S\mu\rho$ where λ, μ are any two vectors (constant), or if $S\lambda\rho = 0$, then what is true of $S\sigma\nabla\sigma$?

11. If $S\sigma\nabla\sigma \neq 0$, show that if F is determined from $S\nabla\sigma\nabla F = -S\sigma\nabla\sigma$ then F is the scalar potential of an irrotational vector τ which added to σ gives a vector σ', $S\sigma'\nabla\sigma' = 0$. Is the equation for F always integrable?

12. The following are vectors whose lines form a congruence of parallel rays $f(\rho)\alpha$, $f(S\alpha\rho)\alpha$, $f(V\alpha\rho)\alpha$, [where f is a scalar function], which are respectively neither solenoidal nor lamellar, lamellar, solenoidal. The case of both demands that $T\sigma =$ constant.

13. Examples of vectors of constant intensity but varying direction are

$$\sigma = aU\rho, \qquad aV\alpha\rho + \alpha\sqrt{(b^2 - a^2V^2\alpha\rho)}.$$

Determine whether these are solenoidal and lamellar.

14. If the lines of a lamellar vector of constant tensor are parallel rays, it is solenoidal. If the lines of a solenoidal vector are parallel straight lines, it is lamellar.

15. An example of vectors whose convergences and curls are equal at all points, and whose tensors are equal at all points of a surface, are

$$\alpha(x + 2yz) + \beta(y + 3zx) + xy\gamma, \quad \text{and} \quad 2yz\alpha + 3zx\beta + \gamma(xy + 2z)$$

and the surface is

$$x^2 + y^2 - z^2 + 6xyz = 0.$$

Therefore vectors are not fully determined when their convergences and curls are given. What additional information is necessary to determine an analytic vector which does not vanish at ∞. Determine a vector which is everywhere solenoidal and lamellar and whose tensor is 12 for $T\rho = \infty$.

16. Show that

$$-\tfrac{1}{6}\nabla^2 q = \lim_{r=0} \text{[average value of } q \text{ over a sphere of radius } r\text{, less the value at the center] divided by } r^2.$$

$$-\tfrac{1}{3}\nabla^2 q = \text{average of } (-S\alpha\nabla)^2 q \text{ in all directions } \alpha.$$

$$-\tfrac{1}{10}\nabla^2 q = \lim_{r=0} \text{[excess of average value of } q \text{ throughout a small sphere over the value at the center] divided by } r^2.$$

17. Show by expansion that

$$\begin{aligned}\sigma(\rho + \delta\rho) &= \sigma(\rho) - S\delta\rho\nabla\cdot\sigma(\rho)\\ &= \nabla_{\delta\rho}[-S\sigma\delta\rho + \tfrac{1}{2}S\delta\rho\nabla_\rho S\sigma\delta\rho] - \tfrac{1}{2}V\delta\rho V\nabla_\rho\sigma\\ &= V\nabla_{\delta\rho}[-\tfrac{1}{2}V\delta\rho\sigma + \tfrac{1}{3}S\delta\rho\nabla_\rho V\delta\rho\sigma] - \tfrac{1}{3}\delta\rho S\nabla_\rho\sigma.\end{aligned}$$

The first expansion expresses σ in the vicinity of ρ in terms of a gradient of a scalar and an infinitesimal rotation. The second expresses σ in the form of a curl and a translation.

18. Show that for any vector σ we have

$$S\nabla(V\nabla'\nabla''S\sigma'\sigma''\sigma/T\sigma^3) = 0,$$

where the accents show on what the ∇ acts, and are removed after the operation of the accented nabla. The unaccented ∇ acts on what is left. (Picard, Traité, Vol. I, p. 136.)

19. If σ_1, σ_2 are two functions of ρ, and $d\sigma_1 = \varphi_1(d\rho)$, $d\sigma_2 = \varphi_2(d\rho)$, show that

$$S\sigma_1\nabla \cdot S\sigma_2\nabla - S\sigma_2\nabla \cdot S\sigma_1\nabla = S(\varphi_1\sigma_2 - \varphi_2\sigma_1)\nabla.$$

16. **Exact Differentials.** If the expression $S\sigma d\rho$ is the differential of a function $u(\rho)$, then it is necessary that $S\sigma d\rho = -Sd\rho\nabla u$, for every value of $d\rho$, which gives

$$\sigma = -\nabla u.$$

When σ is the gradient of a scalar function of $u(\rho)$, u is sometimes called a *force-function*. It is evident at once that

$$V\nabla\sigma = 0, \qquad \text{or} \qquad SV(\nu\nabla)\sigma = 0 \text{ for every } \nu.$$

This is obviously a necessary condition that $S\sigma d\rho$ be an exact differential, that is, be the differential of the same expression, u, for every $d\rho$. It is also sufficient, for if $V\nabla\sigma = 0$, it will be shown below that $\sigma = \nabla u$, and $S\nabla u d\rho = -du$.

In general if $Q(\rho)$ is a linear rational function of ρ, scalar or vector or quaternion, then to be exact, $Q(d\rho)$ must take the form

$$Q(d\rho) = -Sd\rho\nabla \cdot R(\rho) \text{ for every } d\rho.$$

Hence formally we must have the identity

$$Q(\ ) = -S(\ )\nabla \cdot R(\rho).$$

But if we fill the ( ) with the vector form $V\nu\nabla$, we have

$$Q(V\nu\nabla) = 0 \text{ for every } \nu.$$

This may be written in the form

$$Q'V\nabla'(\ ) = 0 \text{ identically.}$$

EXERCISES

1. $V\sigma d\rho$ is exact only when $\sigma = \alpha$ a constant vector. For $V\sigma V\nabla\nu = 0$ for every ν, that is $S\lambda(\nu S\nabla\sigma - \nabla S\sigma\nu) = 0$ for every λ, ν, and for λ perpendicular to ν therefore $S\lambda\nabla S\sigma\nu = 0$, or $Sd\sigma\nu = 0$ for every ν perpendicular to the $d\rho$ that produces $d\sigma$. Again if $\lambda = \nu$,

$$S\nabla\sigma + S\nu\nabla S\sigma\nu = 0,$$

for every ν. Therefore $S\nabla\sigma = 0$ and $S\nu\nabla S\sigma\nu = 0$, or $Sd\sigma\nu = 0$ for every $d\rho$ in the direction of ν. Hence $d\sigma = 0$ for every $d\rho$ and $\sigma = \alpha$ a constant.

2. Examine the expressions

$$S\frac{d\rho}{V\alpha\rho}, \qquad V(V\alpha\rho)d\rho, \qquad V\frac{d\rho}{\rho}.$$

Integrating Factor

If an expression becomes *exact by multiplication* by a scalar function of ρ, let the multiplier be m. Then

$$mQ(V\nabla) = 0,$$

where ∇ operates on m and Q, or

$$QV\nabla m(\ ) + mQV\nabla(\ ) = 0,$$

where ∇ operates on m only in the first term and on Q only in the second. This gives for $S\sigma d\rho$

$$S\sigma\nabla m(\ ) + mS(\ )\nabla\sigma = 0, \qquad \text{or} \qquad V\sigma\nabla m + mV\nabla\sigma = 0.$$

This condition is equivalent however to the condition

$$S\sigma\nabla\sigma = 0.$$

Conversely, when this condition holds, we must have

$$V\nabla\sigma = V\sigma\tau,$$

where τ is arbitrary, hence $S\tau\nabla\sigma = 0$, and $S\sigma\nabla\tau = 0$. But τ is any variable vector conditioned only by being

perpendicular to $V\nabla\sigma$, hence we must have for all such $V\nabla\tau = 0$, or $\sigma = 0$. The latter is obviously out of the question and hence $V\nabla\tau = 0$, that is $\tau = \nabla u$, or we may choose to write it $\tau = \nabla u/u$.

Hence, $V\nabla\sigma + V\nabla u\sigma/u = 0 = V\nabla(u\sigma)$, and $S(u\sigma)d\rho = 0$ is thus proved to be exact.

We may also proceed thus. Since every vector line is the intersection of two surfaces, say $u = 0 = v$, then we can write the curl of σ, which is a vector, in the form

$$V\nabla\sigma = hV\nabla u\nabla v,$$

and if $S\sigma\nabla\sigma = 0$, it follows that we must have σ in the plane of ∇u, ∇v and

$$\sigma = x\nabla u + y\nabla v. \qquad S\sigma d\rho = -xdu - ydv.$$

But also

$$V\nabla\sigma = V\nabla x\nabla u + V\nabla y\nabla v = hV\nabla u\nabla v.$$

Hence

$$S\nabla u\nabla y\nabla v = 0 = S\nabla v\nabla x\nabla u.$$

These are the Jacobians of u, v, x and u, v, y however, and since their vanishing is the condition of functional dependence, it follows that x and y are expressible as functions of u and v. Hence we have

$$x(u, v)du + y(u, v)dv = 0.$$

It is known, however, that this equation in two variables is always integrable by using a multiplier, say g. Therefore $S(g\sigma)d\rho = 0$ is exact for a properly chosen g. Further we see that $g\sigma = -\nabla w$, or that when $S\sigma\nabla\sigma = 0$, $\sigma = m\nabla w$.

If $S\nabla\sigma = 0$ for all points, then we find easily that

$$\sigma = V\nabla\tau.$$

For

$$\sigma = hV\nabla u\nabla v,$$

so that

$$S\nabla\sigma = S\nabla h\nabla u\nabla v = 0$$

and

$$h = h(u, v).$$

Integrate h partially as to u, giving

$$w = \int h du + f(v),$$

then

$$\nabla w = h\nabla u + f_v\nabla v, \qquad V\nabla w\nabla v = hV\nabla u\nabla v = \sigma.$$

Set $\tau = w\nabla v$ or $- v\nabla w$ and we have at once $\sigma = V\nabla\tau$.

It is clear that if we draw two successive surfaces w_1 and w_2 and two successive surfaces v_1 and v_2, since

$$T\nabla w = \frac{\triangle w}{\triangle n_1} \qquad \text{and} \qquad T\nabla v = \frac{\triangle v}{\triangle n_2}$$

and the sides of the parallelogram which is the section of the tube are $\triangle s_2 = \triangle n_1 \csc\theta$, $\triangle s_1 = \triangle n_2 \csc\theta$, and area $= \triangle n_1 \triangle n_2 \csc\theta$, then $T\sigma \times$ area $= \triangle w\triangle v$, and these numbers are constant for the successive surfaces, hence the four surfaces form a tube whose cross-section at every point is inversely as the intensity of σ. For this reason σ is said to be solenoidal or tubular.

If $V\nabla\sigma = 0$ for all points then we must have $\sigma = \nabla u$. For $S\sigma\nabla\sigma = 0$ and $\sigma = g\nabla v$, $V\nabla\sigma = V\nabla g\nabla v$, hence g is a function of v, and we may write

$$\sigma = \nabla u.$$

If $\nabla\sigma = 0$, we must have, since $S\nabla\sigma = 0$, $\sigma = V\nabla\tau$, and since $V\nabla\sigma = 0$, $\sigma = \nabla u$, whence $\nabla^2 u = 0$. Therefore, if $\nabla\sigma = 0$, σ is the gradient of a harmonic function and also the curl of a vector τ, the curl of the curl of τ vanishing. Also if $V\nabla V\nabla\tau = 0$, since we must then have $V\nabla\tau = \nabla v$, and therefore $S\nabla V\nabla\tau = 0 = \nabla^2 v$, we can say that if the curl

of the curl of a vector vanishes it must be such that its curl is the gradient of a harmonic function. Also $Sd\rho\nabla\tau = -dv$. Functions related in the manner of v and τ are very important.

Since in any case $S\nabla V\nabla\sigma = 0$, we must have

$$V\nabla\sigma = V\nabla u\nabla v \qquad \text{or} \qquad V\nabla(\sigma - u\nabla w) = 0,$$

whence

$$\sigma - u\nabla w = \nabla p,$$

so that in any case we may break up a vector σ into the form

$$\sigma = \nabla p + u\nabla w.$$

It follows that $S\sigma\nabla\sigma = S\nabla p\nabla u\nabla w$. If we choose u, w and x as independent variables, we have

$$\nabla p = p_x\nabla x + p_u\nabla u + p_w\nabla w,$$

whence

$$S\sigma\nabla\sigma = p_x S\nabla x\nabla u\nabla w,$$

and we can find p from the integral

$$p = \int S\sigma\nabla\sigma / S\nabla x\nabla u\nabla w\cdot dx.$$

In case $S\sigma\nabla\sigma = 0$, $p =$ constant, and $\sigma = u\nabla w$.

A theorem due to Clebsch is useful, namely that σ can always be put into the form

$$\sigma = \nabla p + V\nabla\tau, \qquad \text{where} \qquad V\nabla\nabla p = 0,\ S\nabla V\nabla\tau = 0,$$

that is, σ can always be considered to be due to the superposition of a solenoidal field upon a lamellar field. We merely have to choose p as a solution of

$$\nabla^2 p = S\nabla\sigma,$$

for we have at once $S\nabla(\sigma - \nabla p) = 0$, and therefore $\sigma - \nabla p = V\nabla\tau$.

This may easily be seen to give us the right to set

$$\sigma = \nabla p + (V\nabla)^n \tau.$$

EXAMPLES. SOLUTIONS OF CERTAIN DIFFERENTIAL FORMS

(1). $S\nabla\sigma = 0$, then $\sigma = V\nabla\tau$, and if $V\nabla\sigma = 0, \sigma = \nabla p$. If $\nabla\sigma = 0$, $\sigma = \nabla h$ where $\nabla^2 h = 0$.

(2). If φ is a linear function dependent upon ρ continuously, and $\varphi\nabla = 0$, $\varphi = \theta V\nabla()$. If $\varphi_0\nabla = 0$,

$$\varphi_0 = V\nabla(\theta_0 V\nabla()),$$

θ, θ_0 are linear functions. For the notation see next chapter.

(3). $V\nabla\varphi() = 0$, $\varphi = -\nabla S\sigma()$. If $\epsilon(V\nabla\varphi()) = 0$, $\varphi = \theta_0 V\nabla() - \nabla S\sigma()$. If $(V\nabla\varphi())_0 = 0, \varphi = p() - \nabla S\sigma()$. $V\nabla\varphi_0 = 0$, $\varphi_0 = -S()\nabla\cdot\nabla p$.

(4). A particular solution of certain forms is given, as follows:

$$S\nabla\sigma = a,\ \sigma = \tfrac{1}{3}a\rho, \qquad V\nabla\sigma = \alpha,\ \sigma = \tfrac{1}{2}V\alpha\rho,$$
$$\nabla p = \alpha,\ p = -S\alpha\rho, \qquad \varphi\nabla = \alpha,\ \varphi = -S\alpha\rho\cdot(),$$
$$V\nabla\varphi() = \theta, \qquad \varphi = -\tfrac{1}{2}V\rho\theta(), \qquad \epsilon(V\nabla\varphi()) = \alpha,$$
$$\varphi = -S\alpha\rho\cdot(), \qquad (V\nabla\varphi)_0 = \theta_0, \qquad \varphi = -\tfrac{1}{2}V\rho\theta_0(),$$
$$V\nabla\varphi = p(), \qquad \varphi = -\tfrac{1}{2}pV\rho() - \nabla S\sigma().$$

EXERCISES

1. Consider the cases $\sigma = i + jf(g(\rho)) + ck$, where f and g have the following values: $f = g, g^2, g^3, \sqrt{g}, \sqrt[3]{g}, g^{-1}, g^{-2}, e^g, \log g, \sin g, \tan g$, and g has the values y/x, $(y - ax)/(ay + x)$, $(bx + y)/(x - by)$, x/y, $-x/y$, $-y/x$, etc., $\sqrt{(x^2 + y^2)} - a$.

2. Consider the vector lines of

$$\sigma = i\cos(3\pi r) + j\sin(3\pi r), \qquad r = \sqrt{(x^2 + y^2)}.$$

3. Consider the significance of $S\cdot U\sigma\nabla U\sigma = 0$; give examples.

4. If $d\sigma = V\tau\, d\rho$ find $V\nabla\sigma$. Likewise if $d\sigma = \alpha d\rho\beta$, $d\sigma = \alpha S\beta d\rho$, $d\sigma = -\rho^2 d\rho$, $d\sigma = V\tau\alpha d\rho$ where τ is a function of ρ.

17. **Groups.** If $\Xi_1, \Xi_2, \cdots, \Xi_n$ are any functions linear

in ∇ but of any degrees in ρ, then they form a transformation group (Lie's) if and only if for any two Ξ_i, Ξ_j,

$$(S\alpha\Xi_i S\beta\Xi_j - S\beta\Xi_j S\alpha\Xi_i) = S(V\alpha\beta)\Theta$$

where Θ is a linear function of $\Xi_1, \Xi_2, \cdots \Xi_n$, and α, β arbitrary vectors. For instance, we have a group in the six formal coefficients of the two vector operators

$$\Xi_1 = -\nabla - \rho S\rho\nabla, \qquad \Xi_2 = -V\rho\nabla,$$

for

$$\begin{aligned} S\alpha\Xi_1 S\beta\Xi_1 - S\beta\Xi_1 S\alpha\Xi_1 &= S\alpha\beta\Xi_2, \\ S\alpha\Xi_2 S\beta\Xi_2 - S\beta\Xi_2 S\alpha\Xi_2 &= -S\alpha\beta\Xi_2, \\ S\alpha\Xi_1 S\beta\Xi_2 - S\beta\Xi_2 S\alpha\Xi_1 &= -S\alpha\beta\Xi_1. \end{aligned}$$

The general condition may be written without α, β:

$$\Xi_i\, S\, \Xi_j' - \Xi_i' S\Xi_j = V\,\Theta\,(),$$

where the accented vector is operated on by the unaccented one.

Integration

18. **Definition.** We define the line integral of a function of ρ, $f(\rho)$, by the expression

$$\int_A^B f(\rho)\varphi(d\rho) = \operatorname*{Lim}_{n=\infty} \Sigma f(\rho_i)\varphi(d\rho_i), \quad i = 1, \cdots, n,$$

where the vectors ρ_i for the n values of i are drawn from the origin to n points chosen along the line from A to B along which the integration is to take place, $\varphi(\sigma)$ is a function which is homogeneous in σ and of first degree, rational or irrational, $d\rho_i = \rho_i - \rho_{i-1}$, and the limit must exist and be the same value for any method of successive subdivision of the line which does not leave any interval finite. Likewise we define a definite integral over an area by the expression

$$\int\int f(\rho)\varphi_2(d_1\rho, d_2\rho) = \operatorname{Lim} \Sigma f(\rho_i)\varphi_2(d_1\rho_i, d_2\rho_i),$$

where φ_2 is a homogeneous function of $d_1\rho_i$ and $d_2\rho_i$, two differentials on the surface at the point ρ_i, and of second degree. A definite integral throughout a volume is similarly defined by

$$\int\int\int f(\rho)\varphi_3(d_1\rho,\ d_2\rho,\ d_3\rho) = \text{Lim}\ \Sigma f(\rho_i)\varphi_3(d_1\rho_i,\ d_2\rho_i,\ d_3\rho_i).$$

For instance, if we consider $f(\rho) = \alpha$, we have for $\int_A^B \alpha d\rho$ along the straight line $\rho = \beta + x\gamma$, $d\rho = dx\cdot\gamma$ and

Lim $\Sigma\alpha dx\cdot\gamma$ from $x = x_0$ to $x = x_1$ is $\alpha\gamma(x_1 - x_0)$, hence

$$\int_{\rho_0}^{\rho_1}\alpha d\rho = \alpha(\rho_1 - \rho_0).$$

The same function along the ellipse $\rho = \beta \cos\theta + \gamma \sin\theta$, where $d\rho = (-\beta\sin\theta + \gamma\cos\theta)d\theta$ has the limit

$$(\alpha\beta\cos\theta + \alpha\gamma\sin\theta)$$

between $\theta = \theta_0$, $\theta = \theta_1$, that is, again $\alpha(\rho_1 - \rho_0)$.

EXAMPLES

(1). $\int_{\rho_0}^{\rho_1} Sd\rho/\rho = \log T\rho_1/\rho_0$, for any path.

(2). $\int_{q_0}^{q_1} - q^{-1}dqq^{-1} = q_1^{-1} - q_0^{-1}$, for any path.

(3). The magnetic force at the origin due to an infinite straight current of direction α and intensity I amperes is $\mathbf{H} = 0.2\cdot I\cdot V\alpha/\rho$, where ρ is the vector perpendicular from the origin to the line. In case then we have a ribbon whose right cross-section by a plane through the origin is any curve, we have the magnetic force due to the ribbon, expressible as a definite integral,

$$\mathbf{H} = 0.2I\int V\alpha Td\rho/\rho.$$

For instance, for a segment of a straight line $\rho = a\beta + x\gamma$, β, γ unit vectors $Td\rho = dx$,

$$\begin{aligned}\mathbf{H} &= 0.2I\int(a\gamma - x\beta)dx/(a^2 + x^2)\\ &= -0.2I\beta\cdot\log(a^2 + x_2^2)/(a^2 + x_1^2)\\ &\qquad + 0.2\cdot I\cdot\gamma(\tan^{-1}x_2/a - \tan^{-1}x_1/a),\\ &= 0.2I\beta\cdot\log OA/OB + 0.2\gamma I\cdot\angle AOB.\end{aligned}$$

(4). Apply the preceding to the case of a skin current in a rectangular conductor of long enough length to be practically infinite, for inside points, and for outside points.

(5). Let the cross-section in (4) be a circle

$$\rho = b\beta - a\beta \cos \theta - a\gamma \sin \theta.$$

Study the particular case when $b = 0$ and the origin is the center.

(6). The area of a plane curve when the origin is in the plane is

$$\tfrac{1}{2} T \int V \rho d\rho.$$

If the curve is not closed this is the area of the sector made by drawing vectors to the ends of the curve. If we calculate the same integral $\frac{1}{2}\int V\rho d\rho$ for a curve not in the plane, or for an origin not in the plane of a curve we will call the result the areal axis of the path, or circuit. This term is due to Koenigs (Jour. de Math., (4) 5 (1889), 323). The projection of this vector on the normal to any plane, gives the projection of the circuit on the plane.

(7). If a cone is immersed in a uniform pressure field (hydrostatic) then the resultant pressure upon its surface is $-\frac{1}{2}\oint V\rho d\rho \cdot P$, where ρ is taken around the directrix curve.

(8). According to the Newtonian law show that the attraction of a straight segment from A to B on a unit point at O is in the direction of the bisector of the angle AOB, and its intensity is $2\mu \sin \frac{1}{2}AOB/c$, where c is the perpendicular from O to the line.

(9). From the preceding results find the attraction of an infinite straight wire, thence of an infinite ribbon, and an infinite prism.

(10). Find the attraction of a cylinder, thence of a solid cylinder.

19. **Integration by Parts.** We may integrate by parts

just as in ordinary problems of calculus. For example,

$$\int_\gamma^\delta V\cdot\alpha d\rho S\beta\rho = \tfrac{1}{2}V\alpha(\delta S\beta\delta - \gamma S\beta\gamma) + \tfrac{1}{2}V\alpha V\beta\int_\gamma^\delta V\rho d\rho,$$

which is found by integrating by parts and then adding to both sides $\int_\gamma^\delta V\cdot\alpha d\rho S\beta\rho$. The integral is thus reduced to an areal integral. In case γ and δ are equal, we have an integral around a loop, indicated by $\oint$.

EXAMPLES

(1). $\int_\gamma^\delta d\rho V\alpha\rho = \frac{1}{2}(\delta V\alpha\delta - \gamma V\alpha\gamma) - \frac{1}{2}V\alpha\int_\gamma^\delta V\rho d\rho + \frac{1}{2}S\alpha\int_\gamma^\delta V\rho d\rho.$

(2). $\int_\gamma^\delta V\cdot V\alpha d\rho V\beta\rho = \frac{1}{2}[\alpha S\beta\int_\gamma^\delta V\rho d\rho + \beta S\cdot\alpha\int_\gamma^\delta V\rho d\rho - \delta S\alpha\beta\delta + \gamma S\alpha\beta\gamma].$

(3). $\int_\gamma^\delta S\cdot V\alpha d\rho V\beta\rho = \frac{1}{2}(S\alpha\delta S\beta\delta - S\alpha\gamma S\beta\gamma - \delta^2 S\alpha\beta - \gamma^2 S\alpha\beta - S\cdot\alpha\beta\int_\gamma^\delta V\rho d\rho).$

(4). $\int_\gamma^\delta V\cdot\alpha d\rho V\beta\rho = \frac{1}{2}(\alpha S\beta\int_\gamma^\delta V\rho d\rho + \beta S\alpha\int_\gamma^\delta V\rho d\rho - \delta S\alpha\beta\delta + \gamma S\alpha\beta\gamma + S\alpha\delta S\beta\delta - S\alpha\gamma S\beta\gamma - \delta^2 S\alpha\beta + \gamma^2 S\alpha\beta - S\alpha\beta\int_\gamma^\delta V\rho d\rho).$

(5). $\int_\gamma^\delta S\alpha\rho S\beta d\rho = \frac{1}{2}[S\alpha\delta S\beta\delta - S\alpha\gamma S\beta\gamma - S\cdot V\alpha\beta\int_\gamma^\delta V\rho d\rho].$

(6). $\int_\gamma^\delta d\rho S\alpha\rho = \frac{1}{2}[\delta S\alpha\delta - \gamma S\alpha\gamma + V\cdot\alpha\int_\gamma^\delta V\rho d\rho].$

(7). $\int_\gamma^\delta V\alpha\rho S\beta d\rho = \frac{1}{2}[V\alpha\delta S\beta\delta - V\alpha\gamma S\beta\gamma - S\alpha\beta\int_\gamma^\delta V\rho d\rho + \beta S\alpha\int_\gamma^\delta V\rho d\rho].$

(8). $\int_\gamma^\delta V\alpha\rho\cdot d\rho = \frac{1}{2}[V\alpha\delta\cdot\delta - V\alpha\gamma\cdot\gamma + \alpha\int_\gamma^\delta V\rho d\rho + S\alpha\int_\gamma^\delta V\rho d\rho].$

(9). $\int_\gamma^\delta \alpha\rho d\rho = \frac{1}{2}[\alpha(\delta^2 - \gamma^2) + 2\alpha\int_\gamma^\delta V\rho d\rho].$

As an example of this formula take the scalar, and notice that the magnetic induction around a wire carrying a

current of value $T\alpha$ amperes, for a circular path σ

$$\mathbf{B} = -\,2\mu V\alpha\rho/a^2.$$

Therefore

$$-\oint 0.2\mu S\alpha\rho d\rho/a^2 = -\,S\oint d\rho\mathbf{B} = -\,0.2\mu a^{-2}S\alpha\oint V\rho d\rho$$
$$= .4T\alpha\mu a^{-2}\pi r^2.$$

For $\mu = 1$, $r = a$, this is $0.4\pi C$. This gives the induction in gausses per turn.

(10). $\int_\gamma^\delta Sd\rho\varphi\rho = \frac{1}{2}[S\delta\varphi\delta - S\gamma\varphi\gamma] + S\epsilon\int_\gamma^\delta V\rho d\rho.$

(11). $\int_\gamma^\delta V\varphi\rho d\rho = \frac{1}{2}[V\gamma\varphi\gamma - V\delta\varphi\delta + \varphi'\int_\gamma^\delta V\rho d\rho + m_1\int_\gamma^\delta V\rho d\rho].$*

(12). $\int_\gamma^\delta \varphi\rho d\rho = \frac{1}{2}[\varphi\delta\cdot\delta - \varphi\gamma\cdot\gamma + S\epsilon\int_\gamma^\delta V\rho d\rho - \varphi'\int_\gamma^\delta V\rho d\rho] - m_1\int_\gamma^\delta V\rho d\rho.$

For any lineolinear form

$$\int_\gamma^\delta Q(\rho, d\rho) = \tfrac{1}{2}[Q(\delta, \delta) - Q(\gamma, \gamma)] + \tfrac{1}{2}\int_\gamma^\delta[Q(\rho, d\rho) - Q(d\rho, \rho)]$$
$$= \tfrac{1}{2}[Q(\delta, \delta) - Q(\gamma, \gamma)] + \tfrac{1}{2}Q'\int_\gamma^\delta V\rho d\rho.$$

(13). State the results for preceding 12 problems for integration around a loop.

(14). Consider forms of second degree in ρ, third degree, etc.

20. **Stokes' Theorem.** We refer now to problem 17, page 189, where we have the value of σ_0, a function of ρ_0, stated for the points in the vicinity of a given fixed point. If we write σ_0 for the value of σ at a given origin 0, its value at a point whose vector is $\delta\rho$ is

$$\sigma = \nabla_{\delta\rho}[-\,S\sigma_0\delta\rho + \tfrac{1}{2}S\delta\rho\nabla S\sigma_0\delta\rho] - \tfrac{1}{2}V\delta\rho V\nabla\sigma_0,$$

where ∇ refers only to σ_0, and gives a value of the curl at

* $m_1(\varphi) = -\,Si\varphi i - Sj\varphi j - Sk\varphi k$. For notation see Chap. IX.

the origin 0. If we multiply by $d\delta\rho$ and take the scalar, we have

$$S\sigma d\delta\rho = d_{\delta\rho}[S\sigma_0\delta\rho - \tfrac{1}{2}S\delta\rho\nabla S\sigma_0\delta\rho] + \tfrac{1}{2}S\delta\rho d\delta\rho V\nabla\sigma_0.$$

Therefore if we integrate this along the curve whose vector radius is $\delta\rho$ we have

$$\begin{aligned}\int_{\delta\rho_1}^{\delta\rho_2} S\sigma_0 d\delta\rho = [S\sigma_0\delta\rho_2 &- S\sigma_0\delta\rho_1 - \tfrac{1}{2}S\delta\rho_2\nabla S\sigma_0\delta\rho_2 \\ &+ \tfrac{1}{2}S\delta\rho_1\nabla S\sigma_0\delta\rho_1] + \tfrac{1}{2}SV\nabla\sigma_0\int V\delta\rho d\delta\rho.\end{aligned}$$

The last expression, however, is the value of

$$S[V\nabla\sigma_0\cdot\text{areal axis of the sector between } \delta\rho_1 \text{ and } \delta\rho_2].$$

Therefore for an infinitesimal circuit we have

$$\oint S\sigma_0 d\delta\rho = S[V\nabla\sigma_0\cdot\text{areal axis of circuit}] = SU\nu V\nabla\sigma_0\cdot dA.$$

$V\nabla\sigma_0$ is the curl of σ at some point inside the loop. If now we combine several circuits which we obtain by subdividing any area, we have for the sum of the line integrals on the left the line integral over the boundary curve of the area in question, and for the expression on the right the sum of the different values of the scalar of the curl of σ multiplied into the unit normals of the areas and the areas themselves or the area integral $\int\int SV\nabla\sigma d_1\rho d_2\rho$. That is, we have for any finite loop, plane or twisted, the formula

$$\oint S\sigma d\rho = \int\int SV\nabla\sigma V d_1\rho d_2\rho.$$

This is called *Stokes' Theorem.* It is assumed in the proof above that there are no discontinuities of σ or $V\nabla\sigma$, although certain kinds of discontinuities can be present. The diaphragm which constitutes the area bounded by the loop is obviously arbitrary, if it is not deformed over a singularity of σ or $V\nabla\sigma$.

It follows that $\int S\sigma d\rho$ along a given path is independent of the path when the expression on the right vanishes for

the possible loops, that is, is zero independently of $d_1\rho$, $d_2\rho$, or that is, $V\nabla\sigma = 0$. This condition is necessary and sufficient.

It follows also that the surface integral of the curl of a vector over a diaphragm of any kind is equal to the circulation of the vector around the boundary of the diaphragm. That is, the flux of the curl is the circuitation around the boundary.

We may generalize the theorem as follows, the expression on the right can be written $\int\int SU\nu V\nabla\sigma\, dA$, where ν is the normal of the surface of the diaphragm and dA is the area element. If now we construct a sum of any number of constant vectors $\alpha_1, \alpha_2, \cdots \alpha_n$ each multiplied by a function of the form $S\sigma_1 d\rho$, $S\sigma_2 d\rho$, $\cdots$ $S\sigma_n d\rho$, we will have a general rational linear vector function of $d\rho$, say $\varphi d\rho$, and arrive at the integral formula

$$\oint \varphi d\rho = \int\int \varphi(VU\nu\nabla)dA,$$

where the ∇ refers now to the functions of ρ implied in φ. This is the vector generalized form of Stokes' theorem.

If the surface is plane, $U\nu$ is a constant, say α, so that for plane paths

$$\oint \varphi d\rho = \int\int \varphi V\nabla\alpha \cdot dA.$$

We may arrive at some interesting theorems by assigning various values to the function φ. For instance, let

$$\varphi d\rho = \sigma d\rho,$$

then

$$\varphi(VU\nu\nabla) = \sigma' VU\nu V\nabla' = -U\nu S\nabla\sigma + \nabla' S\sigma' U\nu + SU\nu\nabla\sigma,$$

whence

$$\int\int S\nabla\sigma \cdot d\nu = \int\int \nabla' S\sigma' d\nu + \oint V\sigma d\rho.$$

If

$$\varphi d\rho = \rho S d\rho\sigma,$$

then

$$\varphi VU\nu\nabla = \rho SU\nu\nabla\sigma - V\sigma U\nu,$$

therefore

$$\int\int V\cdot\sigma d\nu = \int\int \rho Sd\nu\nabla\sigma - \oint \rho S\sigma d\rho.$$

If

$$\varphi d\rho = \rho Vd\rho\sigma,$$
$$\varphi VU\nu\nabla = \rho V(VU\nu\nabla)\sigma - SU\nu\sigma + \sigma U\nu,$$

therefore

$$\int\int \sigma d\nu + S\sigma d\nu = -\int\int \rho V(VU\nu\nabla)\sigma + \oint \rho Vd\rho\sigma,$$

hence

$$2\int\int S\sigma d\nu = -\int\int S\rho(VU\nu\nabla)\sigma + \oint S\rho d\rho\sigma.$$

EXERCISES

1. Investigate the problems of article 19, page 198, as to the application of the theorem.

2. Show that the theorem can be made to apply to a line which is not a loop by joining its ends to the origin, and after applying the theorem to the loop, subtracting the integrals along the radii from 0 to the ends of the line, which can be expressed in terms of dx, along a line. Also consider cases in which the paths follow the characteristic lines of $V\sigma d\rho = 0$.

3. The theorem may be stated thus: the circulation around a path is the total normal flux of the curl of the vector function σ through the loop.

4. If the constant current $I\alpha$ amperes flows in an infinite straight circuit the magnetic force **H** at the point ρ (origin on the axis) is for

$$T\rho \leq a \qquad \mathbf{H} = \tfrac{1}{20}IV\alpha\rho,$$

and for

$$a \leq T\rho \qquad \mathbf{H} = 0.2a^2I/V\alpha\rho,$$

a is the radius of the wire. Then $V\nabla\mathbf{H} = I(\alpha/10)$ inside the wire and equals zero outside. Integrate **H** around various paths and apply Stokes' theorem. In this case the current is a vortex field of intensity $\pi a^2I/10$.

5. If we consider a series of loops each of which surrounds a given tube of vortex lines, it is clear that the circulation around such tube is everywhere the same. If the vector σ defines a velocity field which has a curl, the elementary volumes or particles are rotating, as

we have seen before, the instantaneous axis of rotation being the unit of the curl, and the vector lines of the curl may be compared to wires on which rotating beads are strung. It is known that in a perfect fluid whose density is either constant or a function of the pressure only, and subject to forces having a monodromic potential, the circulation in any circuit through particles moving with the fluid is constant. [Lamb, *Hydrodynamics*, p. 194.] Hence the vortex tubes moving with the fluid (enclosing in a given section the same particles), however small in cross-section, give the same integral of the curl. It follows by passing to an elementary tube that the vortex lines, that is, the lines of curl, move with the fluid, just as if the beads above were to remain always on the same wire, however turbulent the motion. In case the vortex lines return into themselves forming a vortex ring, this leads to the theorem in hydrodynamics that a vortex ring in a perfect fluid is indestructible. It is proved, too, that the same particles always stay in a vortex tube.

6. Show that for $\sigma = \alpha(3S^2\alpha\rho - 2S\beta\rho) + \beta(4S^3\beta\rho - 2S\alpha\rho)$, where $S\alpha\beta = 0$, the integral from the origin to $2\alpha + 2\beta$ is independent of the path. Calculate it for a straight line and for a parabola.

7. The magnetic intensity $\mathbf{H}$, at the point 0, from which the vector ρ is drawn to a filament of wire carrying an infinite straight current in the direction α, of intensity I amperes, is given by

$$\mathbf{H} = 0.2I/V\alpha\rho.$$

Suppose that we have a conductor of any cross-section considered as made up of filaments, find the total magnetic force at 0 due to all the filaments. Notice that

$$\mathbf{H} = 0.2IV\alpha\nabla \log TV\alpha\rho,$$

and that α is the unit normal of the plane cross-section of the conductor. Hence

$$\int\int \mathbf{H}dA = \int\int 0.2IV\alpha\nabla \log TV\alpha\rho dA = \oint 0.2I \log TV\alpha\rho d\rho$$

around the boundary of the cross-section. This can easily be reduced to the ordinary form $0.2I\oint \log r d\rho$. This expression is called a logarithmic potential. If I were a function of the position of the filament in the cross-section, the form of the line-integral would change.

For a circular section we have the results used in problem 4. Consider also a rectangular bar, for inside points and also for outside points.

8. If σ and τ are two vector functions of ρ, we have the theorem

$$SVU\nu\nabla V\sigma\tau = S\tau(VU\nu\nabla)\sigma - S\sigma(VU\nu\nabla)\tau,$$

whence

$$\int\int S\tau(VU\nu\nabla)\sigma = \int\int S\sigma(VU\nu\nabla)\tau + \oint Sd\rho\sigma\tau,$$

for a closed circuit. Show applications when σ or τ or both are solenoidal.

9. Show that

$$\begin{aligned}
\int\int S\cdot d\nu\alpha S\nabla\sigma &= \oint Sd\rho\sigma\alpha + \int\int Sd\nu(S\alpha\nabla)\sigma,\\
\int\int S\cdot\nabla u\sigma dv &= \oint uS\sigma d\rho - \int\int uS\nabla\sigma d\rho,\\
\int\int S\cdot\nabla u\nabla v dv &= \oint uS\nabla v d\rho = -\oint vS\nabla u d\rho,\\
\int_1^2 uS\nabla v d\rho &= [uv]_{\rho 1}^{\rho 2} - \int_1^2 vS\nabla u d\rho.
\end{aligned}$$

10. Prove Koenig's theorems and generalize.

(1) Any area bounded by a loop generates by translation a volume $= -S\alpha\omega$, where ω is the areal axis;

(2) The area for a rotation given by $(\alpha + V\sigma\rho)at$ is $-\int_t S\alpha\omega + \int_t S\sigma\oint V\rho V\rho d\rho$.

21. **Green's Theorem.** The following theorem becomes fundamental in the treatment of surface integrals. Referring to the second form in example 17, page 189, for the expression of a vector in the vicinity of a point, which is

$$\sigma = V\nabla_{\delta\rho}[-\tfrac{1}{2}V\delta\rho\sigma_0 + \tfrac{1}{3}S\delta\rho V\nabla\delta\rho\sigma_0] - \tfrac{1}{3}\delta\rho S\nabla\sigma_0$$

we see that if we multiply by a vector element of surface, $Vd_1\delta\rho d_2\delta\rho$, and take the scalar

$$S\sigma d_1\delta\rho d_2\delta\rho = SU\nu\nabla_{\delta\rho}[]dA - \tfrac{1}{3}S\nabla\sigma_0 Sd_1\delta\rho d_2\delta\rho\delta\rho.$$

If now we integrate over any closed surface the first term on the right gives zero, since the bounding curve has become a mere point, and thus, indicating integration over a closed surface by two $\oint$,

$$\oint\oint S\sigma d_1\delta\rho d_2\delta\rho = -\tfrac{1}{3}S\nabla\sigma_0\oint\oint Sd_1\delta\rho d_2\delta\rho\delta\rho.$$

But the last part of the right hand member is the volume of an elementary triangular pyramid whose base is given by $d_1\delta\rho d_2\delta\rho$. Hence, the integral is the elementary volume of the closed surface, and may be written dv, so that we have for an elementary closed surface

$$\oint\oint S\sigma d_1\delta\rho d_2\delta\rho = \cdot S\nabla\sigma_0 dv.$$

If now we can dissect any volume into elements in which the function has no singularities and sum the entire figure, then pass to the limit as usual, we have the important theorem

$$\oint\oint S\sigma d_1\rho d_2\rho = \int\int\int S\nabla\sigma\, dv.$$

This is called *Green's theorem,* or sometimes *Green's theorem in the first form.* It is usually called Gauss' theorem by German writers, although Gauss' theorem proper was only a particular case and Green's publication antedates Gauss' by several years.

The theorem may be stated thus: the convergence of a vector throughout a given volume is the flux through the bounding surface.

It is evident that we can generalize this theorem as we did Stokes' and thus arrive at the generalized Green's theorem $\oint\oint\Phi\nu dA = \int\int\int\Phi\nabla\, dv.$ ν is the outward unit normal.

The applications are so numerous and so important that they will occupy a considerable space.

The elementary areas and volumes used in proving Stokes' and Green's theorems are often used as integral definitions of convergence or its negative, the divergence, and of curl, rotation, or vortex. For such methods of approach see Joly, Burali-Forti and Marcolongo, and various German texts.

A very obvious corollary is that if $S\nabla\sigma = 0$ then

$$\oint\oint S\sigma d_1\rho d_2\rho = 0.$$

It follows that the flux of any curl through any closed surface is zero. Hence, if the particles of a vortex enter a closed boundary, they must leave it. Therefore, vortex tubes must be either closed or terminate on the boundary wall of the medium in which the vortex is, or else wind about infinitely. We may also state that if $S\nabla\sigma = 0$ the differential expression $S\sigma d_1\rho d_2\rho$ is exact in the sense that

$\int\int S\sigma d_1\rho d_2\rho$ is invariant for different diaphragms bounded by a closed curve, noting the usual restrictions due to singularities.

We proceed to develop some theorems that follow from Green's theorem. Let $\Phi U\nu$ be $-\rho SU\nu\sigma$, then

$$\Phi\nabla = -\rho S\nabla\sigma + \sigma$$

and we have

$$\int\int\int \sigma dv = \int\int\int \rho S\nabla\sigma dv - \oint\oint \rho SU\nu\sigma dA.$$

Let $\Phi U\nu = -\rho VU\nu\sigma$, then $\Phi\nabla = -\rho V\nabla\sigma + 2\sigma$ and

$$\int\int\int \sigma dv = \tfrac{1}{2}\int\int\int \rho V\nabla\sigma dv - \tfrac{1}{2}\oint\oint \rho VU\nu\sigma dA.$$

Let $\Phi U\nu = \rho S\rho U\nu\sigma$, then $\Phi\nabla = \rho S\rho\nabla\sigma + V\rho\sigma$, whence

$$\int\int\int V\rho\sigma\, dv = -\int\int\int \rho S\rho\nabla\sigma\, dv + \oint\oint \rho S\rho U\nu\sigma dA.$$

Let $\Phi U\nu = -\rho V\rho VU\nu\sigma$, then $\Phi\nabla = -\rho V\rho V\nabla\sigma + 3V\rho\sigma$, hence

$$\int\int\int V\rho\sigma dv = \tfrac{1}{3}\int\int\int \rho V\rho V\nabla\sigma dv - \tfrac{1}{3}\oint\oint \rho V\rho VU\nu\sigma dA.$$

Let $\Phi U\nu = S\rho\tau U\nu\sigma$, then $\Phi\nabla = S\rho\tau\nabla\sigma + S\rho\sigma\nabla\tau + S\sigma\tau$, thence

$$\int\int\int S\sigma\tau\, dv = -\int\int\int (S\rho\tau\nabla\sigma + S\sigma\nabla\tau)dv + \oint\oint S\rho\tau U\nu\sigma\, dA.$$

In the first of these if σ has no convergence we have the theorem that the integral of σ, a solenoidal vector, throughout a volume is equal to the integral over the surface of ρ multiplied by the normal component of σ. In the second we have the theorem that if the curl of σ vanishes throughout a volume, so that σ is lamellar in the volume, then the integral of σ throughout the volume is half the integral over the surface of ρ times the tangential component of σ taken at right angles to σ. In the third, if the curl of σ

vanishes then the integral of the moment of σ with regard to the origin is the integral over the surface of $T\rho^2$ times the component along ρ of the negative of the tangential component of σ taken perpendicular to σ, and by the fourth this also equals the surface integral of the component perpendicular to ρ of the negative tangential component of σ taken perpendicular to σ. According to the fifth formula, if a solenoidal vector is multiplied by another and the scalar of the product is integrated throughout a volume, then the integral is the integral of $-S\rho\sigma\nabla\tau$ throughout the volume $+$ the integral of $S\sigma\rho\tau U\nu$ over the surface.

If in the first, second, third, and fourth we set $c\sigma$ for σ, and in the fifth $c\sigma$ for σ and $-\frac{1}{2}\sigma$ for τ, we have from the first and second the expression for λ, the momentum of a moving mass of continuous medium, of density c, and from the third and fourth the moment of momentum, μ, and from the fifth the kinetic energy. If the medium is incompressible, and we set $2\kappa = V\nabla\sigma$, since $S\nabla c\sigma = 0$, then

$$\begin{aligned}
\lambda &= \iiint c\sigma dv \\
&= -\oint\!\!\oint c\rho S U\nu\sigma\, dA + \iiint \rho S\sigma\nabla c\, dv \\
&\qquad\qquad\qquad + \iiint c\rho S\nabla\sigma\, dv \\
&= \iiint \rho c\kappa dv + \tfrac{1}{2}\iiint \rho V\nabla c\sigma dv - \tfrac{1}{2}\oint\!\!\oint c\rho V U\nu\sigma dA.
\end{aligned}$$

$$\begin{aligned}
\mu &= \iiint cV\rho\sigma dv \\
&= \oint\!\!\oint c\rho S\rho U\nu\sigma dA - \iiint c\rho S\rho\nabla\sigma - \iiint \rho S\rho\nabla c\sigma \\
&= \tfrac{2}{3}\iiint c\rho V\rho\kappa + \tfrac{1}{3}\iiint \rho V\rho V\nabla c\sigma dv \\
&\qquad\qquad\qquad - \tfrac{1}{3}\oint\!\!\oint c\rho V\rho V U\nu\sigma dA.
\end{aligned}$$

$$\begin{aligned}
T &= -\tfrac{1}{2}\iiint S\sigma^2 c dv \\
&= -\tfrac{1}{2}\oint\!\!\oint S\rho\sigma U\nu\sigma c\, dA + \iiint \tfrac{1}{2}cS\rho\sigma\nabla\sigma\, dv \\
&\qquad\qquad\qquad + \tfrac{1}{2}\iiint S\rho\sigma\nabla c\sigma\, dv.
\end{aligned}$$

In case c is uniform these become still simpler.

If we set $\sigma = \nabla u$ and $\tau = \nabla w$ in the above formula we

arrive at others for the gradients of scalar functions. The curls will vanish. If further we suppose that u, or w, or both, are harmonic so that the convergences also vanish we have a number of useful theorems.

Other forms of Green's theorem are found by the following methods. Set $\Phi U\nu = uS\nabla wU\nu$, then

$$\Phi\nabla = u\nabla^2 w + S\nabla u\nabla w$$

and we have the second form of Green's theorem at once

$$\iiint S\nabla u\nabla w\,dv = \oint\oint uS\nabla wU\nu\,dA - \iiint u\nabla^2 w\,dv,$$

and from symmetry

$$\iiint S\nabla u\nabla w\,dv = \oint\oint wS\nabla uU\nu\,dA - \iiint w\nabla^2 u\,dv.$$

Subtracting we have

$$\iiint (u\nabla^2 w - w\nabla^2 u)\,dv$$
$$= - \oint\oint (SU\nu[u\nabla w - w\nabla u])dA.$$

22. **Applications.** In the first of these let $u = 1$, then $\iiint \nabla^2 w\,dv = - \oint\oint SU\nu\nabla w dA$. If then w is a harmonic function, the surface integral will vanish, and if $\nabla^2 w = 4\pi\mu$, which is Poisson's equation for potentials of forces varying as the inverse square of the distance, inside the masses, μ being the density of the distribution, then

$$\oint\oint SU\nu\nabla w\,dA = 4\pi M,$$

where M is the total mass of the volume distribution. This is Gauss' theorem, a particular case of Green's. In words, the surface integral of the normal component of the force is $- 4\pi$ times the enclosed mass. The total mass is $1/4\pi$ times the volume integral of the concentration.

In the first formula let $u = 1/T\rho$ and exclude the origin

(a point of discontinuity) by a small sphere, then we have

$$\iiint S\nabla(1/T\rho)\nabla w\, dv$$
$$= \oint\oint dA\; SU\nu\nabla w/T\rho - \iiint dv\; \nabla^2 w/T\rho$$

for the space between the sphere and the bounding surface of the distribution w, and over the two surfaces, the normals pointing *out* of the enclosed space. But for a sphere we have $dA = T\rho^2\, d\omega$ where ω is the solid angle at the center, and $dv = T\rho^2 d\omega dT\rho$. Thus we have

$$\iiint \nabla^2 w/T\rho\; dv$$
$$= \oint\oint S\; dA\; U\nu\nabla w/T\rho - \iiint S\nabla(1/T\rho)\nabla w\; dv$$
$$= \oint\oint S\; dA\; U\nu\nabla w/T\rho - \iiint S\nabla(w\nabla[1/T\rho])dv$$
$$\text{since } \nabla^2 1/T\rho = 0,$$
$$= \oint\oint S\; dA\; U\nu\nabla w/T\rho - \oint\oint S\; dA\; U\nu\; w\; \nabla(1/T\rho)$$
$$= \oint\oint S\; dA\; U\nu\nabla w/T\rho + \oint\oint S\; dA\; U\nu U\rho/T^2\rho w.$$

Now of the integrals on the right let us consider first the surface of the sphere, of small radius $T\rho$. The first integral is then $-\oint\oint SU\rho\nabla w/T\rho\cdot T^2\rho d\omega = -\oint\oint SU\rho\nabla w\cdot T\rho\, d\omega$, and if we suppose that the normal component of ∇w, that is, the component of ∇w along ρ, is everywhere finite, then this integral will vanish with $T\rho$. The second integral for the sphere is $-\oint\oint SU\rho U\rho w T^2\rho d\omega/T^2\rho = -\oint\oint w d\omega$, and the value of w at the origin is w_0, then this integral is $4\pi w_0$ since the total solid angle around a point is 4π. Hence we have

$$\iiint dv\; \nabla^2 w/T\rho = \oint\oint SU\nu(\nabla w/T\rho + wU\rho/T^2\rho)dA$$
$$+ 4\pi w_0$$

and

$$4\pi w_0 = \iiint dv\; \nabla^2 w/T\rho$$
$$- \oint\oint SU\nu(\nabla w/T\rho + wU\rho/T^2\rho)\; dA,$$

where the volume integral is over all the space at which w exists, the origin excluded, and the surface integral is over the bounding surface or surfaces. In words, if we know the value of the concentration of w at every point of space, and the value of the gradient of w and of w at every point of the bounding surfaces at which there is discontinuity, then we can find w itself at every point of space, provided w is finite with its gradient. If $\nabla^2 w$ is of order in ρ not lower than -1 we do not need to exclude the origin, for the integral is $\int\int\int \nabla^2 w \; T\rho d\varphi dT\rho$, and this will vanish with $T\rho$ when $\nabla^2 w$ is not lower in degree than -1.

EXERCISES

1. We shall examine in detail the problem of $w = \text{const.}$ over a given surface, zero over the infinite sphere, $\nabla^2 w = 0$ everywhere, $\nabla w = 0$ on the inside of the sphere, but not zero on the outside. Then for the inside of the sphere the equation reduces to

$$4\pi w_0 = -\oint\oint wSU\nu U\rho/T^2\rho dA = 4\pi w,$$

hence w is constant throughout the sphere and equal to the surface value.

On the outside of the sphere, we have to consider the bounding surfaces to be the sphere and the sphere of infinite radius, so that we have

$$4\pi w_0 = -\oint\oint SdA\; U\nu\nabla w/T\rho - w\oint\oint SdA\; U\nu U\rho/T\rho^2,$$

where the first integral is taken over both surfaces and the second integral is over the given surface only, since $w = 0$ at ∞. The second integral vanishes, however, since it is equal to w times the solid angle of the closed surface at a point exterior to it. If we suppose then that ∇w is 0 at ∞ we have a single integral to evaluate

$$4\pi w_0 = -\oint\oint SdAU\nu\nabla w/T\rho \text{ over the surface.}$$

A simple case is

$$-SU\nu\nabla w = \text{const.} = C.$$

Then

$$4\pi w_0 = C\oint\oint dA/T\rho.$$

The integration of this and of the forms arising from a different assumption as to the normal component of ∇w can be effected by the use of fundamental functions proper to the problem and determined by the boundary conditions, such as Fourier's series, spherical harmonics, and the like. One very simple case is that of the sphere. If we take

the origin at the center of the sphere we have to find the integral

$$\oint\oint dA/T(\rho - \rho_0)$$

where ρ_0 is the vector to the point. Now the solid angle subtended by ρ_0 is given by the integral $-r^{-1}\oint\oint dAS\rho U(\rho - \rho_0)/T^2(\rho - \rho_0)$ and equals 4π or 0, according as the point is inside or outside of the sphere. This integral, however, breaks up easily into two over the surface, the integrands being

$$r^{-1}T^{-1}(\rho - \rho_0) - S\rho_0 U(\rho - \rho_0)/T^2(\rho - \rho_0),$$

but the last term gives 0 or $-4\pi r^2/T\rho_0$, as the point is inside or outside of the sphere. Hence the other term gives

$$\oint\oint dA/T(\rho - \rho_0) = 4\pi r \text{ or } 4\pi r^2/T\rho_0$$

as the point is inside or outside. We find then in this case that

$$w_0 = Cr^2/T\rho_0.$$

If in place of the law above for $-SU\nu\nabla w$, it is equal to $C/T^2(\rho - \rho_0)$ we find that

$$\oint\oint dA/T^3(\rho - \rho_0) = 4\pi r/(r^2 + \rho_0{}^2)$$

or

$$4\pi r^2/(T^3\rho_0 - r^2T\rho_0).$$

Inside

$$-4\pi r + 8\pi r = 4\pi r = -\oint\oint\frac{dA}{T(\rho - \rho_0)} + 2\oint\oint\frac{S\rho_0(\rho - \rho_0)}{T^3(\rho - \rho_0)}dA$$

$$= \int\int dA\,\frac{S(\rho - \rho_0)(\rho + \rho_0)}{T^3(\rho - \rho_0)}.$$

$$dA = 2\pi r^2 \sin\theta\, d\theta = \frac{\pi r}{a}d[a^2 + r^2 - x^2] = \frac{2\pi r}{a}x dx,$$

$$S\frac{\rho_0(\rho - \rho_0)}{T^3(\rho - \rho_0)} = \frac{ax\cos\psi}{x^3} = \frac{a^2 + x^2 - r^2}{2x^3},$$

$$\int\int dAS\frac{\rho_0(\rho - \rho_0)}{T^2(\rho - \rho_0)} = \frac{\pi r}{a}\int_{r-a,\,a-r}^{r+a,\,a+r}\left(\frac{a^2 - r^2}{x^2} + 1\right)dx = 0$$

or

$$\frac{4\pi r^2}{a}.$$

The differentiation of these integrals by using ∇_{ρ_0} as operator under the sign leads to some vector integrals over the surface of the sphere.

2. Show that we have

$$\oint\oint U\nu dA/T(\rho - \rho_0) = \tfrac{4}{3}\pi\rho_0 \qquad \text{or} \qquad \tfrac{4}{3}\pi r^3/T^3\rho_0\cdot\rho_0$$

for inside or outside points of a sphere.

3. Find $\oint\oint dA\,U\nu/T^3(\rho - \rho_0)$ for the sphere.
4. Prove $\oint\oint dA\,T^{-1}(\rho-\beta)T^{-3}(\rho-\alpha) = 4\pi r/[(r^2-a^2)T(\beta-\alpha)]$ or

$$= 4\pi r^2/[a(r^2-a^2)T(r^2\alpha^{-1}+\beta)].$$

5. Consider the case in which the value of w is zero on a surface not at infinity but surrounding the first given surface. We have an example in two concentric spheres which form a condenser. On the inner sphere let w be const. $= W_1$, on the outer let $w = 0$, on the inner let $-SU\nu\nabla w = 0$, inside, $= E_1$, outside, on the outer let $-SU\nu\nabla w = E_2$ on the inside, $= 0$ on the outside.

6. If w is considered with regard to one of its level surfaces, it is constant on the surface, and the integral $-\oint\oint SdA\,U\nu U\rho/T^2\rho w$ becomes for any inside point $4\pi w$, hence we have

$$4\pi w_0 - 4\pi w = \int\int\int dv\nabla^2 w/T\rho - \oint\oint SdA\,U\nu\nabla w/T\rho.$$

If then w is harmonic inside the level surface, it is constant at all points and

$$4\pi(w_0 - w) = -\oint\oint SdA\,U\nu\nabla w/T\rho.$$

But since w_0 is constant as we approach the surface, $\nabla w_0 = 0$, and $\nabla(w - w_0) = 0$, so that $\nabla w = 0$. Hence $w_0 = w$. If w vanishes at ∞ and is everywhere harmonic it equals zero.

7. If two functions w_1, w_2 are harmonic without a given surface, vanish at ∞, and have on the surface values which are constantly in the ratio $\lambda : 1$, λ a constant, then $w_1 = \lambda w_2$.

8. If the surface S_1 is a level for both the functions u and w, as also the surface S_2 inside S_1, and if between S_1 and S_2, u and w are harmonic, then

$$(u - u_1)(w_2 - w_1) = (w - w_1)(u_2 - u_1).$$

For if $w = \varphi(u)$, then $\nabla^2 w = 0 = \varphi''(u)T^2\nabla u$, hence $\varphi(u) = au + b$, etc.

[A scalar point function w is expressible as a function of another scalar function u if and only if $V\nabla w\nabla u = 0$.]

9. Outside a closed surface S, w_1 and w_2 are harmonic and have the same levels. s_1 vanishes at ∞ while w_2 has at ∞ everywhere the constant value C. Then $w_2 = Bw_1 + C$.

For $\nabla w_2 = t\nabla w_1$, $\nabla^2 w_2 = \nabla t\nabla w_1 = 0$, thus $\nabla t = 0$, or $\nabla w_1 = 0$, and $t = B$ or $w_1 =$ const.

10. There cannot be two different functions w_1, w_2 both of which within a given closed surface are harmonic, are continuous with their gradients, are either equal at every point of S or else $SU\nu\nabla w_1 = SU\nu\nabla w_2$ at every point of S while at one point they are equal.

Let $u = w_1 - w_2$, then $\nabla^2 u = 0$, $\nabla u = 0$ on S or else $SU\nu\nabla u = 0$, and at one point $\nabla u = 0$.

11. Given a set of mutually exclusive surfaces, then there cannot be two unequal functions w_1, w_2, which outside all these surfaces are harmonic, continuous with their gradients, vanish at ∞ as $T\rho^{-1}$, their gradients vanishing as $T\rho^{-2}$, and at every point of the surfaces either equal or $SU\nu\nabla w_1 = SU\nu\nabla w_2$.

23. **Solution of Laplace's Equation.** The last problems in the preceding application show that if we wish to invert $\nabla^2 w = 0$, all we need are the boundary conditions, in order to have a unique solution. In case $\nabla^2 u$ is a function of $\rho, f(\rho)$, we can proceed by the method of integral equations to arrive at the integral. However the integral is expressible in the form of a definite integral, as well as a series,

$$w_0 = 1/4\pi\cdot[\int\int\int dv\nabla^2 w/T\rho \\ - \oint\oint SU\nu(\nabla w/T\rho + wU\rho/T^2\rho)dA].$$

The first of these integrals is called the potential and written Pot. Thus for any function of ρ whatever we have

$$\text{Pot } q_0 = \int\int\int q\, dv/T(\rho - \rho_0)$$

where ρ describes the volume and ρ_0 is the point for which Pot q_0 is desired. Let ∇_0 be used to indicate operation as to ρ_0, then we have

$$\begin{aligned}\nabla_0 \text{ Pot } q_0 &= \nabla_0\int\int\int q dv/T(\rho - \rho_0) \\ &= \int\int\int[dvU(\rho - \rho_0)/T^2(\rho - \rho_0)]q \\ &= -\int\int\int\nabla[q/T(\rho - \rho_0)]dv \\ &\qquad + \int\int\int dv\nabla q/T(\rho - \rho_0) \\ &= \text{Pot } \nabla q - \oint\oint dAU\nu q/T(\rho - \rho_0).\end{aligned}$$

If we operate by ∇_0 again, we have

$$\nabla_0^2 \text{ Pot } q_0 = \text{Pot } \nabla^2 q - \oint\oint dA[U\nu\nabla q/T(\rho - \rho_0) \\ + \nabla' U\nu q/T'(\rho - \rho_0)].$$

But the expression on the right is $4\pi q_0$, hence we have the

important theorem

$$\nabla_0^2 \text{ Pot } q_0 = 4\pi q_0.$$

That is, the concentration of a potential is 4π times the function of which we have the potential. In the case of a material distribution of attracting matter, this is Poisson's equation, stating that the concentration of the potential of the density is 4π times the density; that is, given a distribution of attracting masses, they have a potential at any given point, and the concentration of this potential at that point is the density at the point $\div 4\pi$.

The gradient of Pot q_0 was called by Gibbs the Newtonian of q_0, when the function q_0 is a scalar, and if q_0 is a vector, then the curl of its potential is called the Laplacian, and the convergence of its potential is called the Maxwellian of q_0. Thus

$$\text{New } q_0 = \nabla_0 \text{ Pot } P, \qquad \text{Lap } \sigma_0 = V\nabla_0 \text{ Pot } \sigma_0,$$
$$\text{Max } \sigma_0 = S\nabla_0 \text{ Pot } \sigma_0.$$

We have the general inversion formula

$$\begin{aligned}4\pi\nabla_0^{-2}\nabla_0^2 q &= 4\pi q_0\\ &= \iiint \nabla^2 q/T(\rho - \rho_0)dv\\ &\qquad - \oint\oint dA[U\nu Tq/T(\rho - \rho_0)\\ &\qquad\qquad + U(\rho - \rho_0)qU\nu/T^2(\rho - \rho_0)].\end{aligned}$$

This gives us the inverse of the concentration as a potential, plus certain functions arising from the boundary conditions.

We may also define an integral, sometimes useful, called the Helmholtzian,

$$\text{Hlm. } Q = \iiint QT(\rho - \rho_0)dv.$$

Certain double triple integrals have been defined:

$$\text{Pot } (u, v) = \iiiiiint u(\rho_1)v(\rho_2)dv_1dv_2/T(\rho_1 - \rho_2),$$

$$\text{Pot}\ (\xi\eta) = \int\int\int\int\int\int -S\xi\eta\ dv_1 dv_2/T(\rho_1-\rho_2),$$
$$\text{Lap}\ (\xi\eta) = \int\int\int\int\int\int + S\xi\eta(\rho_1-\rho_2)dv_1dv_2/T^3(\rho_1-\rho_2),$$
$$\text{New}\ (v, \xi) = \int\int\int\int\int\int - S\xi_1(\rho_1-\rho_2)v_1dv_1dv_2/T^3(\rho_1-\rho_2),$$
$$\text{Max}\ (\xi, v) = -\int\int\int\int\int\int v_1 S\xi_2(\rho_1-\rho_2)dv_1dv_2/T^3(\rho_1-\rho_2).$$

EXERCISES

1. If $\xi = -\nabla P$ is a field of force or velocity or other vector arising from a scalar function P as its gradient, then

$$P_0 = -\int\int\int S\nabla\xi\ dv/[4\pi T(\rho-\rho_0)] + \oint\oint dA[SU\nu\xi/(4\pi T(\rho-\rho_0)) \\ + PU\nu\nabla T^{-1}(\rho-\rho_0)/4\pi].$$

If P is harmonic the first term vanishes, if $\xi = 0$ the first two vanish.

2. If $\xi = \nabla\sigma$, that is, it is a curl of a solenoidal vector,

$$\sigma_0 = \int\int\int V\nabla\sigma\ dv/[4\pi T(\rho-\rho_0)] - \oint\oint dA[VU\nu\sigma/[4\pi T(\rho-\rho_0)] \\ + U(\rho-\rho_0)\sigma/U\nu[4\pi T^2(\rho-\rho_0)].$$

3. We may, therefore, break up (in an infinity of ways) any vector into two parts, one solenoidal and the other lamellar.

Thus, let $\sigma = \pi + \tau$ where $S\nabla\tau = 0$, and $V\nabla\pi = 0$, then $S\nabla\sigma = S\nabla\pi$ and since $V\nabla\pi = 0$, this may be written $\nabla\pi = S\nabla\sigma$ whence π.

$V\nabla\sigma = V\nabla\tau = \nabla\tau$ whence τ. We have, therefore, from these two

$$4\pi\sigma_0 = \nabla\int\int\int Sdv\nabla\sigma/T(\rho-\rho_0) - \nabla\oint\oint SdA\,U\nu\sigma/T(\rho-\rho_0) \\ +\nabla\oint\oint PSdA\,U\nu\nabla(1/T(\rho-\rho_0) + V\nabla\int\int\int V\nabla\sigma dv/T(\rho-\rho_0) \\ -V\nabla\oint\oint VU\nu\sigma dA/T(\rho-\rho_0) + V\nabla\oint\oint DSU\nu\nabla(T^{-1}(\rho-\rho_0)dA,$$

where P is such that $\nabla^2 P = S\nabla\sigma$ and D such that $\nabla^2 D = V\nabla\sigma$.

3. Another application is found in the second form of Green's theorem. According to the formula

$$\int\int\int(u\nabla^2 w - w\nabla^2 u)\ dv = -\oint\oint(SU\nu[u\nabla w - w\nabla u])dA$$

it is evident that if G is a function such that $\nabla^2 G = 0$, and if, further, G has been chosen so as to satisfy the boundary condition $SU\nu\nabla G = 0$, then the formula becomes

$$\int\int\int G\nabla^2 w\ dv = -\oint\oint SU\nu\nabla wGdA.$$

If then $\nabla^2 w$ is a given function we have the integral equation

$$\oint\oint GSU\nu\nabla w\ dA = -\int\int\int Gf(\rho)dv.$$

Similar considerations enable us to handle other problems.

4. If u and w both satisfy $\nabla^2 f = 0$, then we have Green's Reciprocal Theorem:

$$\oint\oint uSU\nu\nabla w\ dA = \oint\oint wSU\nu\nabla u dA.$$

Thus let

$$w = \frac{1}{T\rho},$$

therefore

$$\oint\!\!\oint \frac{SU\nu\nabla u}{T\rho}\,dA = \oint\!\!\oint uSU\nu\nabla\,\frac{1}{T\rho}\,dA.$$

5. Let $\triangle$ relate to α as ∇ to ρ, then

$$\begin{aligned}\triangle \text{ Pot } Q &= \int\!\!\int\!\!\int Q dv U(\rho - \alpha)/T^2(\rho - \alpha)\\ &= \int\!\!\int\!\!\int \nabla\,(Q/T(\rho - \alpha))dv + \int\!\!\int\!\!\int dv \nabla Q/T(\rho - \alpha)\\ &= \text{Pot } \nabla Q - \int\!\!\int dAU\nu Q/T(\rho - \alpha).\end{aligned}$$

If $Q = 0$ on the surface, the surface integral $= 0$.

New P = Pot $V - \oint\!\!\oint U\nu P\, dA/T(\rho - \alpha) = \triangle$ Pot P when Pot exists.

Lap $\sigma = V$ Pot $\nabla\sigma - \oint\!\!\oint VU\nu\sigma dA/T(\rho - \alpha) = V\triangle$ Pot σ when Pot exists.

Max $\sigma = S$ Pot $\nabla\sigma - \oint\!\!\oint SU\nu\sigma dA/T(\rho - \alpha) = S\triangle$ Pot σ when Pot exists.

$$\triangle^2 \text{ Pot } Q = \text{Pot } \nabla^2 Q - \oint\!\!\oint U\nu\nabla Q dA/T(\rho - \alpha) + \oint\!\!\oint dA\nabla_1[U\nu Q/T_1(\rho - \alpha)].$$

If $Q = 0$ on the surface, that is, if Q has no surface of discontinuity,

$$\begin{aligned}\triangle^2 \text{ Pot } Q &= \text{Pot } \nabla^2 Q,\\ \triangle \text{ New } P &= \triangle^2 \text{ Pot } P,\\ \triangle \text{ Lap } \sigma &= \triangle V \triangle \text{ Pot } \sigma,\\ \triangle \text{ Max } \sigma &= \triangle S \triangle \text{ Pot } \sigma.\end{aligned}$$

6. If β is a function of the time t, then

$$\begin{aligned}\beta = -\frac{1}{4\pi}\Big[&\int\!\!\int\!\!\int \frac{1}{r}\Big(V\nabla V\nabla\beta + b^2\frac{\partial^2\beta}{\partial t^2}\Big]_{t+br}\\ &+ \nabla \int\!\!\int\!\!\int \frac{1}{r} S\nabla\beta_{t+br} - \nabla \oint\!\!\oint \frac{1}{r} S\beta_{t+br} d\nu\\ &+ V\nabla \oint\!\!\oint \frac{1}{r} Vd\nu\, \beta_{t+br} - \oint\!\!\oint Vd\nu\, V\nabla\beta_{t+br}\end{aligned}$$

where the subscript means $t + br$ is put for t after the operations on β have occurred.

CHAPTER IX

THE LINEAR VECTOR FUNCTION

1. **Definition.** If there is a vector σ which is an integral rational function φ of the vector ρ,

$$\sigma = \varphi \cdot \rho,$$

and if in this function we substitute for ρ a scalar multiple $t\rho$ of ρ, then we call the vector function a *linear vector function* if σ becomes $t\sigma$ under this substitution. It is also called a *dyadic*.

The function φ transforms the vector ρ, which may be in any direction, into the vector σ, which may not in every case be able to take all directions. If $\rho = \alpha$, then we have $\varphi\rho = \varphi\alpha$, and φ as an operator has a value at every point in space. We may, in fact, look upon φ as a space transformation that deforms space by a shift in its points leaving invariant the origin and the surface at infinity. In the case of a straight line

$$V\alpha\rho = \beta, \qquad \text{or} \qquad \rho = x\alpha + \alpha^{-1}\beta,$$

we see that the operation of φ on all its vectors gives

$$\sigma = x\varphi\alpha + \varphi V\alpha^{-1}\beta,$$

and this is a straight line whose equation is

$$V\varphi\alpha\sigma = V\varphi\alpha\varphi V\alpha^{-1}\beta,$$

which will later be shown to reduce to a function of β only, $\varphi\beta$. Hence φ converts straight lines into straight lines. The lines α for which $V\alpha\varphi\alpha = 0$, remain parallel

to their original direction, others change direction. Again if we consider the plane $S\cdot\alpha\beta\rho = 0$ or

$$\rho = x\alpha + y\beta, \qquad \sigma = x\varphi\alpha + y\varphi\beta,$$

so that

$$S\sigma\varphi\alpha\varphi\beta = 0.$$

Hence planes through the origin, and likewise all planes, are converted into planes. These will be parallel to their original direction if $V\alpha\beta = uV\varphi\alpha\varphi\beta$, or

$$VV\alpha\beta V\varphi\alpha\varphi\beta = 0 = S\alpha\varphi\alpha\varphi\beta = S\beta\varphi\alpha\varphi\beta = S\alpha\beta\varphi\alpha = S\alpha\beta\varphi\beta.$$

Now $V\alpha\beta$ is normal to the plane, and β is any vector in the plane, and $\varphi\beta$ by the equation is normal to $V\alpha\beta$, hence $\varphi\beta = v\alpha + w\beta$ for all vectors β in the plane.

Since $\varphi 0 = 0$, the function leaves the origin invariant. Consequently the lines and planes through the origin that remain parallel to themselves are invariant as lines and planes. These lines we will call the invariant lines of φ, and the planes the invariant planes of φ.

2. **Invariant Lines.** In order to ascertain what lines are invariant we solve the equation

$$V\alpha\varphi\alpha = 0, \qquad \text{or} \qquad \varphi\alpha = g\alpha,$$

that is

$$(\varphi - g)\alpha = 0.$$

First we write α in the form

$$\alpha S\lambda\mu\nu = \lambda S\mu\nu\alpha + \mu S\nu\lambda\alpha + \nu S\lambda\mu\alpha,$$

where λ, μ, ν are any three noncoplanar vectors. Then we have at once

$$(\varphi - g)\lambda S\mu\nu\alpha + (\varphi - g)\mu S\nu\lambda\alpha + (\varphi - g)\nu S\lambda\mu\alpha = 0.$$

But this means that we must have for any three noncoplanar vectors λ, μ, ν

$$S(\varphi - g)\lambda(\varphi - g)\mu(\varphi - g)\nu = 0$$
$$= g^3 S\lambda\mu\nu - g^2(S\lambda\mu\varphi\nu + S\lambda\varphi\mu\nu + S\varphi\lambda\mu\nu)$$
$$+ g(S\lambda\varphi\mu\varphi\nu + S\lambda\mu\varphi\nu + S\varphi\lambda\varphi\mu\nu) - S\varphi\lambda\varphi\mu\varphi\nu,$$

an equation to determine g, which we shall write

$$g^3 - m_1 g^2 + m_2 g - m_3 = 0,$$

called the *latent equation* of φ, where we have set

$$m_1 = (S\lambda\mu\varphi\nu + S\lambda\varphi\mu\nu + S\varphi\lambda\mu\nu)/S\lambda\mu\nu,$$
$$m_2 = (S\lambda\varphi\mu\varphi\nu + S\varphi\lambda\mu\varphi\nu + S\varphi\lambda\varphi\mu\nu)/S\lambda\mu\nu,$$
$$m_3 = S\varphi\lambda\varphi\mu\varphi\nu/S\lambda\mu\nu.$$

These expressions are called the *nonrotational scalar invariants* of φ. That they are invariant is easily seen by substituting $\lambda' + v\mu$ for λ. The resulting form is precisely the same for λ', μ, ν, and from the symmetry involved this means that for λ, μ, ν we can substitute any other three noncoplanar vectors, and arrive at the same values for m_1, m_2, m_3. It is obvious that m_3 is the ratio in which the volume of the parallelepiped λ, μ, ν is altered. If $m_3 = 0$ one or more of the roots of the cubic are zero. The number of zero roots is called the *vacuity* of φ. If is obvious that the latent cubic has either one or three real roots.

3. **General Equation.** We prove now a fundamental equation due to Hamilton. Starting with φ we iterate the function on any vector, as ρ, writing the successive results thus

$$\rho,\ \varphi\rho,\ \varphi\varphi\rho = \varphi^2\rho, \qquad \varphi\varphi\varphi\rho = \varphi\varphi^2\rho = \varphi^3\rho, \qquad \cdots.$$

We have then for any three vectors λ, μ, ν that are not coplanar

$$\begin{aligned}
S\lambda\mu\nu(\varphi^3\rho - m_1\varphi^2\rho) &= \varphi^2(\varphi\rho - m_1\rho)S\lambda\mu\nu \\
&= \varphi^2[\varphi\lambda S\mu\nu\rho + \cdots - \rho S\mu\nu\varphi\lambda - \cdots] \\
&= -\varphi^2[V\cdot V\mu\nu V\varphi\lambda\rho + \cdots] \\
&= \varphi^2[V\cdot V\varphi\lambda\rho V\mu\nu + \cdots] \\
&= \varphi[\varphi\lambda S\nu\varphi\mu\rho + \varphi\mu S\lambda\varphi\nu\rho + \varphi\nu S\mu\varphi\lambda\rho \\
&\quad - \varphi\lambda S\mu\varphi\nu\rho - \varphi\mu S\nu\varphi\lambda\rho \\
&\qquad - \varphi\nu S\lambda\varphi\mu\rho].
\end{aligned}$$

Adding to this result $S\lambda\mu\nu\cdot m_2\varphi\rho$, we have

$$\begin{aligned}
&S\lambda\mu\nu(\varphi^3\rho - m_1\varphi^2\rho + m_2\varphi\rho) \\
&\quad = \varphi[\lambda S\varphi\mu\varphi\nu\rho + \mu S\varphi\nu\varphi\lambda\rho + \nu S\varphi\lambda\varphi\mu\rho] = \rho S\varphi\lambda\varphi\mu\varphi\nu.
\end{aligned}$$

Subtracting $S\lambda\mu\nu\cdot m_3\rho$ from both sides and dropping the nonvanishing factor $S\lambda\mu\nu$, we have the *Hamilton cubic* for φ

$$\varphi^3\rho - m_1\varphi^2\rho + m_2\varphi\rho - m_3\rho = 0.$$

This cubic holds for all vectors ρ, and hence, may be written symbolically

$$\varphi^3 - m_1\varphi^2 + m_2\varphi - m_3 = 0$$

identically. This is also called the *general equation for* φ. It is the same equation so far as form goes as the latent equation. Hence we may write it in the form

$$(\varphi - g_1)(\varphi - g_2)(\varphi - g_3) = 0.$$

In other words, the successive application of these three operators to any vector will identically annul it.

We scarcely need to mention that the three operators written here are commutative and associative, since this follows at once from the definition of linear vector operator, and of its powers.

It is to be noted, too, that φ may satisfy an equation of lower degree. This, in case there is one, will be called the *characteristic equation* of φ. Since φ must satisfy its general

equation, the process of highest common divisor applied to the two will give us an equation which φ satisfies also, and as this cannot by hypothesis be lower than the characteristic equation in degree and must divide it, it is the characteristic equation. Hence the factors of the characteristic equation are included among those of the general equation. We proceed now to prove that the general equation can have no factors different from the factors of the characteristic equation.

(1) Let the characteristic equation be

$$(\varphi - g)\rho = 0$$

for every vector; then assuming any λ, μ, ν, we find easily for the latent equation

$$x^3 - 3gx^2 + 3g^2x - g^3 = 0,$$

so that the general equation is

$$(\varphi - g)^3 = 0.$$

In this case

$$\varphi = [g\lambda S\mu\nu() + g\mu S\nu\lambda() + g\nu S\lambda\mu()]/S\lambda\mu\nu,$$

where λ, μ, ν are given for a given φ.

(2) Let the characteristic equation be

$$(\varphi - g_1)(\varphi - g_2)\rho = 0,$$

then by hypothesis, there is at least one vector α for which we have

$$(\varphi - g_1)\alpha \neq 0,$$

and at least one β for which

$$(\varphi - g_2)\beta \neq 0.$$

Let us take then

$$(\varphi - g_1)\alpha = \lambda, \qquad (\varphi - g_2)\beta = \mu.$$

Then

$$(\varphi - g_2)\lambda = 0, \qquad (\varphi - g_1)\mu = 0.$$

Hence, we cannot have λ and μ parallel, else $g_1 = g_2$, which we assume is not the case, since from

$$(\varphi - g_2)U\lambda = 0, \qquad (\varphi - g_1)U\mu = 0,$$

we have

$$g_2U\lambda = g_1U\mu, \qquad \text{and} \qquad g_2 = g_1,$$

if λ is parallel to μ, that is if $U\lambda$ would $= U\mu$.

There is still a third direction independent of λ and μ, say ν. Let

$$\varphi\nu = a\nu + b\mu + c\lambda.$$

Then we have

$$(\varphi - g_1)\nu = (a - g_1)\nu + b\mu + c\lambda.$$

Since

$$(\varphi - g_2)(\varphi - g_1)\nu = 0,$$

$$\begin{aligned}(a - g_1)(\varphi - g_2)\nu - b(g_2 - g_1)\mu &= 0\\ &= (a - g_1)(a - g_2)\nu + b(a - g_2)\mu + c(a - g_1)\lambda.\end{aligned}$$

We must have, therefore, either

$$a = g_1 \qquad \text{and} \qquad b = 0,$$

or

$$a = g_2 \qquad \text{and} \qquad c = 0.$$

As the numbering of the roots is immaterial, let us take $a = g_1$, $b = 0$, then

$$\varphi\nu = g_1\nu + c\lambda, \qquad \varphi\lambda = g_2\lambda, \qquad \varphi\mu = g_1\mu,$$

We notice that if $c \neq 0$, we can choose $\nu' = \nu - (c/g_2)\lambda$, whence $\varphi\nu' = g_1\nu'$ and we could therefore take $c = 0$. Hence

$$g^3 - g^2(2g_1 + g_2) + g(2g_1g_2 + g_1^2) - g_1^2g_2 = 0,$$

$$\varphi = [g_1\mu S\lambda\nu() + g_1\nu S\lambda\mu() + g_2\lambda S\mu\nu()]/S\lambda\mu\nu,$$

and the general equation is

$$(\varphi - g_1)^2(\varphi - g_2) = 0.$$

(3) Let the characteristic equation be

$$(\varphi - g)^2\rho = 0.$$

Then there is one direction λ for which we have

$$\varphi\lambda = g\lambda,$$

and there may be other directions for which the same is true. There is at least one direction μ such that

$$(\varphi - g)\mu = \lambda.$$

We have, therefore,

$$\varphi\mu = g\mu + \lambda, \qquad \varphi\lambda = g\lambda.$$

Let now ν be a third independent direction, then we have

$$\begin{aligned}\varphi\nu &= a\nu + b\mu + c\lambda,\\ (\varphi - g)\nu &= (a - g)\nu + b\mu + c\lambda,\\ (\varphi - g)^2\nu &= 0 = (a - g)^2\nu + b(a - g)\mu + [b + c(a - g)]\lambda.\end{aligned}$$

Therefore, we have $a = g$, $b = 0$, $\varphi\nu = g\nu + c\lambda$ and $\varphi(\nu - c\mu) = g(\nu - c\mu) = g\nu'$, and the general equation

$$(\varphi - g)^3 = 0,$$
$$\varphi = g + \lambda S\nu\lambda()/S\lambda\mu\nu.$$

We are now in a position to say that the general equation has exactly the same factors as the characteristic equation. Further we can state as a theorem the following:

(a) *If the characteristic equation is of first degree,*

$$(\varphi - g_1)\rho = 0,$$

then every vector is converted into g_1 times that vector, by the operation of φ.

(*b*) *If the characteristic equation is of the form*

$$(\varphi - g_1)(\varphi - g_2) = 0,$$

then there is one direction λ *such that* $\varphi\lambda = g_2\lambda$, *while for every vector in a given plane of the form* $x\mu + y\nu$ *we have*

$$(\varphi - g_1)(x\mu + y\nu) = 0.$$

Hence φ *multiplies by* g_1 *every vector in the plane of* μ, ν, *and by* g_2 *all vectors in the direction* λ.

(*c*) *If the characteristic equation is*

$$(\varphi - g_1)^2 = 0,$$

there is a direction such that

$$\varphi\lambda = g_1\lambda,$$

and a given plane such that for every vector in it $x\mu + y\nu$ *we have*

$$(\varphi - g_1)(x\mu + y\nu) = h\lambda.$$

If $(\varphi - g_1)\mu = v\lambda$, $(\varphi - g_1)\nu = w\lambda$, we may set

$$\mu' = \mu - \frac{v}{w}\,\nu,$$

giving $\varphi\mu = g_1\mu$. *Therefore* φ *extends all vectors in the ratio* g_1, *and shears all components parallel to* ν *in the direction* λ.

4. **Nondegenerate Equations.** We have left to consider the three cases

$$\begin{aligned}(\varphi - g_1)(\varphi - g_2)(\varphi - g_3) &= 0,\\ (\varphi - g_1)^2(\varphi - g_2) &= 0,\\ (\varphi - g_1)^3 &= 0.\end{aligned}$$

In the last case we see easily that there is a set of unit vectors λ, μ, ν such that

$$\varphi\lambda = g_1\lambda + \mu a,$$
$$\varphi\mu = g_1\mu + \nu b,$$
$$\varphi\nu = g_1\nu.$$

Hence we see that

$$\varphi(x\lambda + y\mu + z\nu) = g_1(x\lambda + y\mu + z\nu) + ax\mu + by\nu$$
$$= g_1(x\lambda + y\mu + z\nu) + a(x\mu + y\nu) + (b - a)y\nu,$$
$$\varphi(x\mu + y\nu) = g_1(x\mu + y\nu) + bx\nu,$$
$$\varphi = g_1 + [a\mu S\mu\nu() + b\nu S\nu\lambda()]/S\lambda\mu\nu.$$

Therefore φ extends all vectors in the ratio g_1, shears all vectors λ in the direction of μ, and all vectors μ in the direction ν.

In the first case we see that there is at least one vector ρ such that

$$(\varphi - g_2)(\varphi - g_3)\rho = \lambda,$$

where

$$\varphi\lambda = g_1\lambda.$$

Likewise there are vectors that lead to μ and ν where $\varphi\mu = g_2\mu$, $\varphi\nu = g_3\nu$. These are independent, and therefore if we consider any vector

$$\rho = x\lambda + y\mu + z\nu,$$

we have

$$\varphi\rho = xg_1\lambda + yg_2\mu + zg_3\nu,$$
$$\varphi = [g_1\lambda S\mu\nu() + g_2\mu S\nu\lambda() + g_3\nu S\lambda\mu()]/S\lambda\mu\nu.$$

Evidently we can find λ, μ, ν by operating on all vectors necessary in order to arrive at nonvanishing results by

$$(\varphi - g_2)(\varphi - g_3), \quad (\varphi - g_1)(\varphi - g_3), \quad (\varphi - g_1)(\varphi - g_2)$$

respectively.

In the second case, we see in a similar manner that there

are three vectors such that

$$\varphi\lambda = g_1\lambda + \mu, \qquad \varphi\mu = g_1\mu, \qquad \varphi\nu = g_2\nu,$$
$$\varphi = [g_1(\lambda S\mu\nu() + \mu S\nu\lambda() + g_2\nu S\lambda\mu() + \mu S\mu\nu()]/S\lambda\mu\nu.$$

5. **Summary.** We may now summarize these results in the following theorem, which is of highest importance.

Every linear vector function satisfies a general cubic, and may also satisfy an equation of lower degree called the characteristic equation. If the equation of lowest degree is the cubic, then it may have three distinct latent roots, in which case there corresponds to each root a distinct invariant line through the origin, any vector in each of the three directions being extended in a given ratio equal to the corresponding root; or it may have two equal roots, in which case there corresponds to the unequal root an invariant line, and to the multiple root an invariant plane containing an invariant line, every vector in the plane being multiplied by the root and then affected by a shear of its points parallel to the invariant line in the plane; or there may be three equal roots, in which case there is an invariant line, a plane through this line, every line of the plane through the origin being multiplied by the root and its points sheared parallel to the invariant line, and finally every line in space not in this plane is multiplied by the root and its points sheared parallel to the invariant plane. In case the function satisfies a reduced equation which is a quadratic, this quadratic may have unequal roots, in which case there is an invariant line corresponding to one root and an invariant plane corresponding to the other, any line in the plane through the origin being multiplied by the corresponding root; or there may be two equal roots, in which case there is an invariant plane such that every line in the plane is multiplied by the root and every vector not in the plane is multiplied by the root and its points displaced parallel to an invariant line. In case

the reduced equation is of the first degree, every line is an invariant line, all vectors being extended in a fixed ratio. Where there are displacements, they are proportional to the distance from the origin, and the region displaced is called a shear region.

Hence φ takes the following forms in which g_1, g_2, g_3 may be equal, or any two may be equal:

I. $[g_1\alpha S\beta\gamma() + g_2\beta S\gamma\alpha() + g_3\gamma S\alpha\beta()]/S\alpha\beta\gamma$; reduced equations for $g_1 = g_2$ or $g_1 = g_2 = g_3$;

II. $[g_1\alpha S\beta\gamma() + g_1\beta S\gamma\alpha() + g_2\gamma S\alpha\beta() + a\beta S\beta\gamma()]/S\alpha\beta\gamma$; reduced equation for $g_1 = g_2$, or if $a = 0$;

III. $g + [(a\beta + c\gamma)S\beta\gamma() + b\gamma S\gamma\alpha()]/S\alpha\beta\gamma$, reduced if $a = 0 = c$, or $a = 0 = b = c$.

EXAMPLES

(1). Let $\varphi\rho = V\cdot\alpha\rho\beta$, where $S\alpha\beta \neq 0$. Take $\lambda = \alpha$, $\mu = \beta$, $\nu = V\alpha\beta$, then we find with little trouble

$$m_1 = -S\alpha\beta, \qquad m_2 = -\alpha^2\beta^2, \qquad m_3 = \alpha^2\beta^2 S\alpha\beta,$$

and the characteristic equation of φ,

$$(\varphi + S\alpha\beta)(\varphi - T\alpha\beta)(\varphi + T\alpha\beta) = 0.$$

Hence there are three invariant lines in general, and operating on ρ by $(\varphi + S\alpha\beta)(\varphi - T\alpha\beta)$, we find the invariant line corresponding,

$$(\varphi + S\alpha\beta)\rho = \alpha S\beta\rho + \beta S\alpha\rho,$$

$$\begin{aligned}(\varphi - T\alpha\beta)(\varphi + S\alpha\beta)\rho \\ &= \alpha^2\beta S\beta\rho + \beta^2\alpha S\alpha\rho - \alpha T\alpha\beta S\beta\rho - \beta T\alpha\beta S\alpha\rho \\ &= -(T\alpha S\beta\rho + T\beta S\alpha\rho)(U\alpha + U\beta)T\alpha\beta.\end{aligned}$$

Hence the invariant line corresponding to the root $T\alpha\beta$ is $U\alpha + U\beta$. The other two are

$$U\alpha - U\beta \qquad \text{and} \qquad UV\alpha\beta.$$

(2). Let $\varphi\rho = V\alpha\beta\rho$.

(3). Let $\varphi\rho = g_2\alpha S\beta\gamma\rho + g_1(\beta S\gamma\alpha\rho + \gamma S\alpha\beta\rho) + h\beta S\alpha\beta\rho$.

(4). Let $\varphi\rho = g\rho + (h\beta + l\gamma)S\beta\gamma\rho + r\beta S\gamma\alpha\rho$.

(5). Let $\varphi\rho = V\epsilon\rho$.

6. **Solution of** $\varphi\rho = \alpha$. It is obvious that when φ satisfies the general equation

$$\varphi^3 - m_1\varphi^2 + m_2\varphi - m_3 = 0, \qquad m_3 \neq 0,$$

then the vector

$$m_3\varphi^{-1}\rho = (m_2 - m_1\varphi + \varphi^2)\rho.$$

For if we take the φ function of this vector, we have an identity for all values of ρ. Also this vector is unique, for if a vector α had to be added to the left side, or could be added to the left side, then it would have to satisfy the equation $\varphi\alpha = 0$. But if $m_3 \neq 0$, there is no vector satisfying this equation, for this equation would lead to a zero root for φ. Hence, if $\varphi\rho = \lambda$, $m_3\rho = m_2\lambda - m_1\varphi\lambda + \varphi^2\lambda$, which solves the equation.

If φ satisfies the general equation

$$\varphi_3{}^3 - m_1\varphi^2 + m_2\varphi = 0, \qquad m_2 \neq 0,$$

then we have one and only one zero root of the latent equation, and corresponding to it a unique vector for which $\varphi\alpha = 0$, and if $\varphi\rho = \lambda$,

$$m_2\rho = x\alpha + (m_1\varphi - \varphi^2)\rho = x\alpha + m_1\lambda - \varphi\lambda.$$

If φ satisfies the cubic

$$\varphi^3 - m_1\varphi^2 = 0, \qquad m_1 \neq 0,$$

the vacuity is two, and we have two cases according as there is not a reduced equation, or a reduced equation exists

of the form $\varphi^2 - m_1\varphi = 0$. In either case the other root is m_1. There is a corresponding invariant line λ, and if the vector α is such that $\varphi\alpha = 0$, then we have in the two cases a vector β such that respectively $\varphi\beta = \alpha$, or $\varphi\beta = 0$. Hence, if $\varphi\rho = \gamma$, we must have in the two cases

$$\gamma = x\lambda + y\alpha, \qquad \text{or} \qquad \gamma = x\lambda.$$

Otherwise the equation is impossible. Hence

$$m_1\rho = x\lambda + z\alpha + y\beta = \gamma + u\alpha + y\beta,$$

where $\varphi\beta = \alpha$, $\varphi\alpha = 0$, or where $\varphi\beta = 0 = \varphi\alpha$.

If φ satisfies the cubic

$$\varphi^3 = 0,$$

and no reduced equation, there are three vectors (of which β and γ are not unique) such that $\varphi\gamma = \beta$, $\varphi\beta = \alpha$, $\varphi\alpha = 0$, and then $\varphi\rho = \lambda$, we must have $\lambda = x\alpha + y\beta$, where ρ is any vector of the form

$$\rho = z\alpha + x\beta + y\gamma.$$

If $\varphi^2 = 0$, and no lower degree vanishes, then

$$\varphi(x\beta + y\gamma) = \alpha, \qquad \varphi\alpha = 0, \qquad \text{and} \qquad \lambda = u\alpha.$$

If $\varphi = 0$, there is no solution except for $\varphi\rho = 0$, where ρ may be any vector.

7. **Zero Roots.** It is evident that if one root is zero, then the region $\varphi\lambda$ where λ is any vector will give us the other roots. For instance let $\varphi\rho = V\epsilon\rho$. Then if $\mu = V\epsilon\lambda$,

$$\varphi\mu = \lambda\epsilon^2 - \epsilon S\epsilon\lambda, \qquad \varphi^2\mu = \epsilon^2\mu,$$

and the other two roots are $\pm\sqrt{-1}\cdot T\epsilon$.

If two roots are zero, then φ^2 on any vector will give the invariant region of the other root. For instance, let

$\varphi\rho = \alpha S\beta\gamma\rho$, then $\alpha S\beta\gamma\alpha S\beta\gamma\rho = \varphi^2\rho$. Hence $\varphi\alpha = \alpha S\alpha\beta\gamma$ gives the other root as $S\alpha\beta\gamma$ and its invariant line α.

In case a root is not zero, but is g_1, if it is of multiplicity one, then $\varphi - g_1$ operating upon any vector will give the region of the other root, or roots. If it is of multiplicity two, then we use $(\varphi - g_1)^2$ on any vector.

8. **Transverse.** We define now a linear vector operator related to φ, and sometimes equal to φ, which we shall indicate by φ', and call the conjugate of φ, or transverse of φ, and define by the equation

$$S\lambda\varphi\mu = S\mu\varphi'\lambda \quad \text{for all} \quad \lambda, \mu.$$

For example, if $\varphi\rho = V\alpha\rho\beta$, then $S\lambda\varphi\rho = S\lambda\alpha\rho\beta = S\rho\beta\lambda\alpha$, and $\varphi' = V\beta()\alpha = \varphi$, if $\varphi\rho = V\epsilon\rho$, $\varphi'\rho = - V\epsilon\rho$; if $\varphi\rho = \alpha S\beta\rho$, $\varphi'\rho = \beta S\alpha\rho$.

If α is an invariant line of φ, $\varphi\alpha = g\alpha$, then for every β

$$S\beta\varphi\alpha = gS\alpha\beta = S\alpha\varphi'\beta,$$

or

$$S\alpha(\varphi' - g)\beta = 0,$$

that is α is perpendicular to the region not annulled by $\varphi' - g$, that is invariant for $\varphi' - g$. If we consider that from the definition we have equally

$$S\lambda\varphi^2\mu = S\mu\varphi'^2\lambda, \qquad S\lambda\varphi^3\mu = S\mu\varphi'^3\lambda,$$

it is clear that φ and φ' have the same characteristic equation and the same general equation. They can differ only in their invariant regions if at all. If then the roots are all distinct, it is evident that the invariant line α of φ, is normal to the two invariant lines of φ' corresponding to the other two roots, hence each invariant line of φ is normal to the two of φ' corresponding to the other roots, and conversely. If now the characteristic equation is the general equation,

so that each function satisfies only the general equation, let there be two equal roots, g, whose shear region gives

$$\varphi\alpha = g\alpha + \beta, \qquad \varphi\beta = g\beta, \qquad \text{let} \qquad \varphi\gamma = g_1\gamma.$$

Then

$$S\alpha\varphi'\rho = gS\alpha\rho + S\beta\rho, \quad S\beta\varphi'\rho = gS\beta\rho, \quad S\gamma\varphi'\rho = g_1S\gamma\rho,$$
$$S\alpha\beta\gamma\cdot\varphi'\rho = g(V\beta\gamma S\alpha\rho + V\gamma\alpha S\beta\rho) + V\beta\gamma S\beta\rho + g_1V\alpha\beta S\gamma\rho.$$

Therefore corresponding to the root g_1, φ' has the invariant line $V\alpha\beta$, and to the root g, the invariant line $V\beta\gamma$. Further φ' converts $V\gamma\alpha$ into $gV\gamma\alpha + V\beta\gamma$.

Hence the invariant line of g_1 for φ' is normal to the shear region of g, and the shear region of g for φ' is normal to the invariant line of g_1 for φ, but the invariant line of g for φ' is normal further to the shear direction of g for φ, and the shear direction of φ' for g is normal to the invariant line of φ for g.

In case there are three equal roots, and no reduced equation, we have

$$\varphi\alpha = g\alpha + \beta, \qquad \varphi\beta = g\beta + \gamma, \qquad \varphi\gamma = g\gamma,$$

so that

$$S\alpha\varphi'\rho = gS\alpha\rho + S\beta\rho, \qquad S\beta\varphi'\rho = gS\beta\rho + S\gamma\rho,$$
$$S\gamma\varphi'\rho = gS\gamma\rho,$$
$$S\alpha\beta\gamma\cdot\varphi'\rho = \rho gS\alpha\beta\gamma + V\beta\gamma S\beta\rho + V\gamma\alpha S\gamma\rho.$$

Hence, the invariant line of φ' is $V\beta\gamma$, its first shear line $V\gamma\alpha$, and second shear line $V\alpha\beta$.

In case there is a reduced equation with two distinct roots, we have

$$\varphi(x\alpha + y\beta) = g(x\alpha + y\beta), \qquad \varphi\gamma = g_1\gamma,$$
$$S\alpha\varphi'\rho = gS\alpha\rho, \qquad S\beta\varphi'\rho = gS\beta\rho, \qquad S\gamma\varphi'\rho = g_1S\gamma\rho,$$
$$S\alpha\beta\gamma\cdot\varphi'\rho = gV\beta\gamma S\alpha\rho + gV\gamma\alpha S\beta\rho + g_1V\alpha\beta S\gamma\rho.$$

Hence, the invariant line of φ' corresponding to g_1 is normal to the invariant plane of g for φ, corresponding to g there is an invariant plane normal to the invariant line of g_1 for φ. Every line in the plane through the origin is invariant.

In case the reduced equation has two equal roots, then

$$\varphi\alpha = g\alpha + \beta, \qquad \varphi\beta = g\beta, \qquad \varphi\gamma = g\gamma,$$
$$S\alpha\varphi'\rho = gS\alpha\rho + S\beta\rho, \qquad S\gamma\varphi'\rho = gS\gamma\rho, \qquad S\beta\varphi'\rho = gS\beta\rho,$$
$$S\alpha\beta\gamma\cdot\varphi'\rho = g\rho + S\beta\rho\cdot(V\beta\gamma).$$

Corresponding to g, we have then two invariant lines: $V\alpha\beta$, which is perpendicular to the shear plane of φ; $V\beta\gamma$, which is perpendicular to the non-shear region of g and to the shear direction of g; also the shear direction of φ' is $V\beta\gamma$, so that the shear region of φ' is determined by $V\gamma\alpha$ and $V\beta\gamma$, and is therefore perpendicular to γ.

The three forms of φ' are

I. $\varphi' = [g_1V\beta\gamma S\alpha() + g_2V\gamma\alpha S\beta() + g_3V\alpha\beta S\gamma()]/S\alpha\beta\gamma;$

II. $\varphi' = [g_1V\beta\gamma S\alpha() + g_1V\gamma\alpha S\beta() + aV\beta\gamma S\beta()$
$\qquad + g_2V\alpha\beta S\gamma()]/S\alpha\beta\gamma;$

III. $\varphi' = g + [aV\beta\gamma S\beta() + bV\gamma\alpha S\gamma() + cV\beta\gamma S\gamma()]/S\alpha\beta\gamma.$

We may summarize these results in the theorem:

The invariant regions of φ' corresponding to the distinct roots are normal to the corresponding regions of the other roots for φ. In case there are repeated roots, if there is a plane every line of which through the origin is invariant, then every line of the corresponding plane will also be invariant, but if there is a plane with an invariant line and a shear direction in it, the first invariant line of the conjugate will be perpendicular to the shear direction and to the second invariant line of φ, and the shear direction of the conjugate will be perpendicular to the invariant lines of φ;

while finally, if there is an invariant line, a first shear direction, and a second shear direction, then the invariant line of the conjugate will be perpendicular to the invariant line and the first shear direction of φ, *the first shear direction will be perpendicular to the invariant line and the second shear direction of* φ, *and the second shear direction will be perpendicular to the two shear directions of* φ. *Let* α, β, γ *define the various directions* $\bar{\alpha} = V\beta\gamma/S\alpha\beta\gamma$, $\bar{\beta} = V\gamma\alpha/S\alpha\beta\gamma$, $\bar{\gamma} = V\alpha\beta/S\alpha\beta\gamma$, *then we have*

$$\begin{Bmatrix} \varphi\ = g_1\alpha S\bar{\alpha} + g_2\beta S\bar{\beta} + g_3\gamma S\bar{\gamma} \\ \varphi' = g_1\bar{\alpha} S\alpha + g_2\bar{\beta} S\beta + g_3\bar{\gamma} S\gamma \end{Bmatrix}$$

or

$$\begin{Bmatrix} g_1\alpha S\bar{\alpha} + g_1\beta S\bar{\beta} + a\beta S\bar{\alpha} + g_2\gamma S\bar{\gamma} \\ g_1\bar{\alpha} S\alpha + g_1\bar{\beta} S\beta + a\bar{\alpha} S\beta + g_2\bar{\gamma} S\gamma \end{Bmatrix}$$

or

$$\begin{Bmatrix} g + (a\beta + c\gamma)S\bar{\alpha} + b\gamma S\bar{\beta} \\ g + a\bar{\alpha} S\beta + (b\bar{\beta} + c\bar{\alpha})S\gamma \end{Bmatrix}.$$

9. **Self Transverse.** It is evident now that $\varphi = \varphi'$ only when there are no shear regions, if we limit ourselves to real vectors, and further the invariant lines must be perpendicular or if two are not perpendicular, then every vector in their plane must be an invariant, and even in this case the invariants may be taken perpendicular. Hence every real self-transverse linear vector operator may be reduced to the form

$$\varphi\rho = -\alpha S\alpha\rho g_1 - \beta S\beta\rho g_2 - \gamma S\gamma\rho g_3,$$

where $\alpha\ \beta\ \gamma$ form a trirectangular system, and where the roots g may be equal.

Conversely, when $\varphi = \varphi'$, the roots are real, provided that we have only real vectors in the system, for if a root has the form $g + ih$, where $i = \sqrt{-1}$, then if the invariant

line for this root be $\lambda + i\mu$, where λ and μ are real, we have

$$\varphi(\lambda + i\mu) = (g + ih)(\lambda + i\mu) = g\lambda - h\mu + i(h\lambda + g\mu)$$
$$= \varphi\lambda + i\varphi\mu.$$

Therefore

$$\varphi\lambda = g\lambda - h\mu, \qquad \varphi\mu = h\lambda + g\mu,$$

and

$$S\mu\varphi\lambda = gS\lambda\mu - h\mu^2 = S\lambda\varphi\mu = h\lambda^2 + gS\lambda\mu.$$

Thus we must have

$$h\lambda^2 + h\mu^2 = 0.$$

It follows that $h = 0$.

Of course the roots may be real without φ being self-transverse.

An important theorem is that $\varphi\varphi'$ and $\varphi'\varphi$ are self-transverse. For

$$S\rho\varphi\varphi'\sigma = S\sigma\varphi\varphi'\rho, \qquad S\rho\varphi'\varphi\sigma = S\sigma\varphi'\varphi\rho.$$

EXERCISE

Find expressions for $\varphi\varphi'$ and $\varphi'\varphi$ in terms of α, β, γ, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$.

10. **Chi of φ.** We define now two very important functions related to φ and always derivable from it. First

$$\chi_\varphi = m_1 - \varphi,$$

so that

$$\begin{aligned} S\alpha\beta\gamma\cdot\chi_\varphi\rho &= \rho S\alpha\beta\varphi\gamma + \rho S\beta\gamma\varphi\alpha + \rho S\gamma\alpha\varphi\beta - \varphi\alpha S\beta\gamma\rho \\ &\qquad - \varphi\beta S\gamma\alpha\rho - \varphi\gamma S\alpha\beta\rho \\ &= VV\alpha\beta V\varphi\gamma\rho + \cdots \\ &= \alpha S\rho(\beta\varphi\gamma - \gamma\varphi\beta) + \cdots. \end{aligned}$$

The other function is indicated by ψ_φ or by $\chi_{\varphi\varphi}$ and defined

$$\begin{aligned} \psi_\varphi &= m_2 - m_1\varphi + \varphi^2 = m_2 - \varphi\chi_\varphi, \\ S\alpha\beta\gamma\cdot\psi_\varphi\rho &= \rho S\alpha\varphi\beta\varphi\gamma + \cdots - \varphi\alpha S\rho(\beta\varphi\gamma - \gamma\varphi\beta) \\ &= \alpha S\rho\varphi\beta\varphi\gamma + \beta S\rho\varphi\gamma\varphi\alpha + \gamma S\rho\varphi\alpha\varphi\beta. \end{aligned}$$

We have at once from these formulae the following important forms for $V\lambda\mu$,

$$\begin{aligned}\chi_\varphi V\lambda\mu &= [\alpha SV\lambda\mu(V\beta\varphi\gamma - V\gamma\varphi\beta)\cdots]/S\alpha\beta\gamma \\ &= [\alpha S(V\varphi'\lambda\mu - V\lambda\varphi'\mu)V\beta\gamma + \cdots]/S\alpha\beta\gamma \\ &= V\varphi'\lambda\mu + V\lambda\varphi'\mu.\end{aligned}$$

Whence we have also

$$\begin{aligned}\varphi V\lambda\mu &= m_1 V\lambda\mu - V\lambda\varphi'\mu - V\varphi'\lambda\mu, \\ \psi_\varphi V\lambda\mu &= [\alpha SV\lambda\mu V\varphi\beta\varphi\gamma + \cdots]/S\alpha\beta\gamma \\ &= V\varphi'\lambda\varphi'\mu.\end{aligned}$$

Since it 's evident that

$$\chi_{\varphi'} = \chi_\varphi', \qquad \text{and} \qquad \psi_{\varphi'} = \psi_\varphi',$$

we have at once

$$\begin{aligned}\chi'_\varphi V\lambda\mu &= V\varphi\lambda\mu + V\lambda\varphi\mu \\ \psi'_\varphi V\lambda\mu &= V\varphi\lambda\varphi\mu.\end{aligned}$$

The two expressions on the right are thus shown to be functions of $V\lambda\mu$.

It is evident that as multipliers of ρ

$$\begin{aligned}m_1 &= \varphi + \chi = \varphi' + \chi', \\ m_2 &= \varphi\chi + \psi = \varphi'\chi' + \psi', \\ m_3 &= \varphi\psi = \varphi'\psi'.\end{aligned}$$

EXERCISES

1. If $\varphi = \alpha_1 S\beta_1() + \alpha_2 S\beta_2() + \alpha_3 S\beta_3()$, show that

$$\begin{aligned}\varphi' &= \beta_1 S\alpha_1() + \beta_2 S\alpha_2() + \beta_3 S\alpha_3(), \\ \chi &= \Sigma V\beta_1 V\alpha_1(), \\ \psi &= -\Sigma V\beta_1\beta_2 SV\alpha_1\alpha_2(),\end{aligned}$$

$$m_1 = \Sigma S\alpha_1\beta_1, \qquad m_2 = -\Sigma SV\alpha_1\alpha_2 V\beta_1\beta_2, \qquad m_3 = -S\alpha_1\alpha_2\alpha_3 S\beta_1\beta_2\beta_3,$$

$$\begin{aligned}\chi' &= \Sigma V\alpha_1 V\beta_1(), \\ \psi' &= -\Sigma V\alpha_1\alpha_2 SV\beta_1\beta_2().\end{aligned}$$

2. Show that the irrotational invariants of χ and ψ are $m_1(\chi) = 2m_1$, $m_2(\chi) = m_1^2 + m_2$, $m_3(\chi) = m_1m_2 - m_3$; $m_1(\psi) = m_2$, $m_2(\psi) = m_1m_3$, $m_3(\psi) = m_3^2$.

3. For any linear vector function φ, and its powers φ^2, φ^3, $\cdots$, we have

$$m_1(\varphi^2) = m_1^2 - 2m_2, \qquad m_2(\varphi^2) = m_2^2 - 2m_1m_3, \qquad m_3(\varphi^2) = m_3^2.$$
$$m_1(\varphi^3) = m_1^3 - 3m_1m_2 + 3m_3, \quad m_2(\varphi^3) = 3m_1m_2m_3 - m_2^3 - 3m_3^2,$$
$$m_3(\varphi^3) = m_3^3.$$
$$m_1(\varphi^4) = m_1^3 - 4m_1^2m_2 + 2m_2^2 + 4m_1m_3$$
$$m_2(\varphi^4) = m_2^4 - 4m_1m_2^2m_3 + 2m_1^2m_3^2 + 4m_2m_3^2, \qquad m_3(\varphi^4) = m_3^4.$$

4. Show that for the function $\varphi + c$, where c is a scalar multiplier,

$$m_1(\varphi + c) = m_1(\varphi) + 3c, \qquad m_2(\varphi + c) = m_2(\varphi) + 2m_1(\varphi)c + 3c^2,$$
$$m_3(\varphi + c) = m_3(\varphi) + cm_2(\varphi) + c^2m_1(\varphi) + c^3.$$

5. Study functions of the form $x\psi + y\chi + z$.

6. $\varphi' V\varphi\lambda\varphi\mu = m_3 V\lambda\mu$; $\varphi'(V\lambda\varphi\mu - V\mu\varphi\lambda) = m_2 V\lambda\mu - V\varphi\lambda\varphi\mu$.

7. $\psi(a\varphi) = a^2\psi(\varphi)$; $\psi(\varphi_1\varphi_2) = \psi(\varphi_2)\cdot\psi(\varphi_1)$.

8. $\psi(a) = a^2$, $\psi[V\alpha()] = -\alpha S\alpha()$, $\psi(-\beta S\alpha) = 0$.
$\psi(-g_1 iSi - g_2 jSj - g_3 kSk) = -g_2g_3 iSi - g_3g_1 jSj - g_1g_2 kSk$.

9. $\psi(\alpha_1 S\beta_1 + \alpha_2 S\beta_2 + \alpha_3 S\beta_3)$
$$= -V\beta_1\beta_2 SV\alpha_1\gamma_2 - V\beta_2\beta_3 SV\alpha_2\alpha_3 - V\beta_3\beta_1 SV\alpha_3\alpha_1.$$

10. For any two operators φ, θ,

$$m_1(\varphi\theta) = m_1(\theta\varphi), \qquad m_2(\varphi\theta) = m_2(\theta\varphi), \qquad m_3(\varphi\theta) = m_3(\theta\varphi).$$
$$m_1(\varphi\theta) = m_1(\varphi)m_1(\theta) + m_2(\varphi) + m_2(\theta) - m_2(\theta + \varphi).$$
$$m_2(\varphi\theta) = m_2(\theta)m_2(\varphi) + m_3(\varphi)\cdot m_1(\theta) + m_3(\theta)\cdot m_1(\varphi) - m_2[\psi'(\varphi) + \psi'(\theta)].$$
$$m_3(\varphi\theta) = m_3(\varphi)\cdot m_3(\theta).$$
$$m_1(\varphi + \theta) = m_1(\varphi) + m_1(\theta).$$
$$m_2(\varphi + \theta) = m_2(\varphi) + m_2(\theta) + m_1(\theta)\cdot m_1(\varphi) - m_1(\varphi\theta).$$
$$m_3(\varphi + \theta) = m_3(\varphi) + m_3(\theta) + m_1[\varphi'\psi'(\theta) + \theta'\psi'(\varphi)].$$

11. χ can have the three forms:

I. $(g_2 + g_3)\alpha S\bar{\alpha} + (g_3 + g_1)\beta S\bar{\beta} + (g_1 + g_2)\gamma S\bar{\gamma}$;

II. $(g_1 + g_2)\alpha S\bar{\alpha} + (g_1 + g_2)\beta S\bar{\beta} + 2g_1\gamma S\gamma + a\beta S\alpha$;

III. $2g - (a\beta + c\gamma)S\bar{\alpha} - b\gamma S\bar{\beta}$.

The operator χ is the rotor dyadic of Jaumann.

12. The forms of ψ for the three types are

I. $g_2g_3\alpha S\bar{\alpha} + g_3g_1\beta S\bar{\beta} + g_1g_2\gamma S\bar{\gamma}$;

II. $g_1g_2\alpha S\bar{\alpha} + g_2g_1\beta S\bar{\beta} + g_1^2\gamma S\bar{\gamma} - ag_2\beta S\bar{\alpha}$;

III. $g^2 - [ag\beta + (ab - gc)\gamma]S\bar{\alpha} - bg\gamma.S\bar{\beta}$.

13. An operator called the *deviator* is defined by Schouten,* and is for the three forms as follows:

I. $(\frac{2}{3}g_1 - g_2 - g_3)\alpha S\bar{\alpha} + (\frac{2}{3}g_2 - g_3 - g_1)\beta S\bar{\beta} + (\frac{2}{3}g_3 - g_1 - g_2)\gamma S\bar{\gamma};$

II. $(-\frac{1}{3}g_1 - g_2)(\alpha S\bar{\alpha} + \beta S\bar{\beta}) + (\frac{2}{3}g_2 - 2g_1)\gamma S\bar{\gamma} + a\beta S\bar{\alpha};$

III. $(a\beta + c\gamma)S\bar{\alpha} + b\gamma S\bar{\beta}.$

It is $V\varphi = \varphi - S\varphi$, where $S(\varphi) = \frac{1}{3}m_1$.

14. Show that if $F(\lambda, \mu) = -F(\mu, \lambda)$ then

$$F(\lambda, \mu) = C(\lambda, \mu)Q.V\lambda\mu,$$

where C is symmetric in λ, μ and Q is a quaternion function of $V\lambda\mu$.

11. We derive from φ and φ' the two functions

$$\tfrac{1}{2}(\varphi + \varphi') = \varphi_0, \qquad \tfrac{1}{2}(\varphi - \varphi') = V\epsilon().$$

That there is a vector ϵ satisfying this last equation, and which is invariant, is easily shown. For if we form $m_3(\varphi - \varphi')$, we find that

$$\begin{aligned}
&S(\varphi - \varphi')\lambda(\varphi - \varphi')\mu(\varphi - \varphi')\nu \\
&\quad = S\varphi\lambda\varphi\mu\varphi\nu - \Sigma S\varphi\lambda\varphi'\mu\varphi'\nu - \Sigma S\varphi\lambda\varphi\,\mu\varphi'\nu - S\varphi'\lambda\varphi'\mu\varphi'\nu \\
&\quad = S\lambda\mu\nu(m_3 - m_3 + m_1(\psi'\varphi) - m_1(\psi\varphi')).
\end{aligned}$$

But it is easy to see that this expression vanishes identically, for the first two terms cancel, and if φ_1, φ_2 are any two linear vector functions, we have

$$\begin{aligned}
&S^2\lambda\mu\nu\cdot m_1(\varphi_1\varphi_2) \\
&\quad = S\mu\nu\varphi_1\lambda S\mu\nu\varphi_2\lambda + S\mu\nu\varphi_1\mu S\nu\lambda\varphi_2\lambda + S\mu\nu\varphi_1\nu S\lambda\mu\varphi_2\lambda \\
&\qquad + S\nu\lambda\varphi_1\lambda S\mu\nu\varphi_2\mu + S\nu\lambda\varphi_1\mu S\nu\lambda\varphi_2\mu + S\nu\lambda\varphi_1\nu S\lambda\mu\varphi_2\mu \\
&\qquad + S\lambda\mu\varphi_1\lambda S\mu\nu\varphi_2\nu + S\lambda\mu\varphi_1\mu S\tau\varphi_2\nu + S\lambda\mu\varphi_1\nu S\lambda\mu\varphi_2\nu \\
&\quad = S^2\lambda\mu\nu\cdot m_1(\varphi_2\varphi_1).
\end{aligned}$$

Hence we may under m_1 permute cyclically the vector functions. Again after this has been done we may take the conjugate. Hence the expression above vanishes, and there is a zero root in all cases for $\varphi - \varphi'$. Further we may always write

* Grundlagen der Vector- und Affinor-Analysis, p. 64.

$$\begin{aligned} S\lambda\mu\nu\varphi\rho &= \varphi\lambda S\mu\nu\rho + \cdots \\ S\lambda\mu\nu\cdot\varphi'\rho &= V\mu\nu S\lambda\varphi'\rho + \cdots \\ &= V\mu\nu S\varphi\lambda\rho + \cdots. \end{aligned}$$

Hence we have

$$S\lambda\mu\nu\cdot(\varphi - \varphi')\rho = V\rho V(V\mu\nu)\varphi\lambda + \cdots.$$

From this we have $2\epsilon S\lambda\mu\nu = V\varphi\lambda V\mu\nu + \cdots$ for every noncoplanar λ, μ, ν.

The function φ_0 is evidently self-transverse, and the conjugate of $V\epsilon()$ is $-V\epsilon()$. It is easy to show that

$$2\varphi\epsilon S\lambda\mu\nu = -V\lambda V\varphi\mu\varphi\nu - \cdots.$$

The expressions $T\epsilon$, $T\varphi\epsilon$, and $S\epsilon\varphi\epsilon$ are scalar invariants of φ, and these three may be called the *rotational invariants.* In terms of them and the other three scalar invariants all scalar invariants of φ or φ' may be expressed.

If there are three distinct roots, g_1, g_2, g_3, and the corresponding invariant unit vectors are γ_1, γ_2, γ_3, we may set these for λ, μ, ν, and thus

$$\begin{aligned} 2\epsilon S\gamma_1\gamma_2\gamma_3 &= g_1V\gamma_1V\gamma_2\gamma_3 + g_2V\gamma_2V\gamma_3\gamma_1 + g_3V\gamma_3V\gamma_1\gamma_2 \\ &= (g_2 - g_3)\gamma_1S\gamma_2\gamma_3 + (g_3 - g_1)\gamma_2S\gamma_3\gamma_2 \\ &\qquad + (g_1 - g_2)\gamma_3S\gamma_1\gamma_2. \\ 2\varphi\epsilon S\gamma_1\gamma_2\gamma_3 &= -g_2g_3V\gamma_1V\gamma_2\gamma_3 - g_3g_1V\gamma_2V\gamma_3\gamma_1 \\ &\qquad - g_1g_2V\gamma_3V\gamma_1\gamma_2. \end{aligned}$$

In case two roots are equal and $\varphi\alpha \doteq g_1\alpha + \beta h^2$, $\varphi\beta = g_1\beta$, $\varphi\gamma = g_2\gamma$, we have

$$2\epsilon S\alpha\beta\gamma = (g_2 - g_1)V\gamma V\alpha\beta + V\beta V\beta\gamma h.$$

In case three roots are equal, $\varphi\alpha = g\alpha + h\beta$, $\varphi\beta = g\beta + l\gamma$, $\varphi\gamma = g\gamma$

$$2\epsilon S\alpha\beta\gamma = h\beta V\beta\gamma + l\gamma V\gamma\alpha.$$

It is evident, therefore, that if the roots are distinct and

the axes perpendicular two and two, that $\epsilon = 0$; if two roots are equal and the invariant line of the other root is perpendicular to the plane of the equal roots, then it is the direction of ϵ; and if the three roots are equal, and if the invariant line is perpendicular to the two shear directions, then ϵ is in the plane of the invariant line and the second shear.

12. **Vanishing Invariants.** The vanishing of the scalar invariants of φ leads to some interesting theorems.

If $m_1 = 0$, there is an infinite set of trihedrals which are transformed by φ into trihedrals whose edges are in the faces of the original trihedral. If φ transforms any trihedral in this manner, $m_1 = 0$, and there is an infinite set of trihedrals so transformed.

We choose λ, μ, ν for the edges of the vertices, and if $\varphi\lambda$ is coplanar with μ, γ, $\varphi\mu$ with ν, λ, and $\varphi\nu$ with λ, μ, the invariant $m_1 = 0$. If $m_1 = 0$, we choose λ, μ, arbitrarily, and determine ν from $S\varphi\lambda\mu\nu = 0 = S\lambda\varphi\mu\nu$. Then also $S\lambda\mu\varphi\nu = 0$.

The invariant m_2 vanishes if φ transforms a trihedral into another whose faces pass through the edges of the first. The converse holds for any infinity of trihedrals.

EXERCISES

1. Show that if α, β, γ form a trirectangular system

$$m_1 = -S\alpha\varphi\alpha - S\beta\varphi\beta - S\gamma\varphi\gamma$$

and is invariant for all trirectangular systems,

$$\begin{aligned} m_2(\varphi\varphi') &= T^2\varphi\alpha + T^2\varphi\beta + T^2\varphi\gamma, \\ T^2\varphi(\lambda) &= S^2\lambda\varphi\alpha + S^2\lambda\varphi\beta + S^2\lambda\varphi\gamma. \end{aligned}$$

2. Study the functions for the ellipsoid and the two hyperboloids

$$-\varphi = a^{-2}\alpha S\alpha \pm b^{-2}\beta S\beta \pm c^{-2}\gamma S\gamma.$$

3 Study the functions

$$\Sigma m V\alpha V()\alpha, \qquad \varphi + V\alpha V()\alpha, \qquad \alpha^{-1}V\alpha\varphi(),$$
$$\varphi - V\beta V\alpha(), \qquad V\cdot\varphi V\alpha()\cdot\beta.$$

4. Show that

$$\nabla \varphi\rho = 2\epsilon - m_1, \qquad \nabla \psi\rho = -2\varphi\epsilon - m_2,$$
$$\nabla S\rho\varphi\rho = -2\varphi_0\rho, \qquad \nabla V\rho\varphi\rho = 2S\epsilon\rho + m_1\rho - 3\varphi\rho,$$

wherein φ is a constant function. Hence $\varphi_0\rho$ may always be represented as a gradient of a scalar, $S\epsilon\rho$ as a convergence of a vector, and $m_1\rho - 3\varphi\rho$ (deviation) as a curl. We may consider also that m_1 is a convergence and ϵ is a curl, m_2 a convergence and $\varphi\epsilon$ a curl.

5. An orthogonal function is defined to be one such that

$$\varphi\varphi' = 1.$$

Show that an orthogonal function can be reduced to the form

$$\varphi = ()\cos\theta - \sin\theta \cdot V()\beta = (1 \mp \cos\theta)\beta S\beta() = \beta^{+\theta/\pi}()\beta^{-\theta/\pi}$$

or $-\beta^{(\theta/\pi)+1}()\beta^{-(\theta/\pi)-1}$ which is a rotation about the axis β through the angle $-\theta$, or such a rotation followed by reflection in the plane normal to β.

6. Study the operator $\varphi^{1/2}$.

7. Show that

$$m_1(\varphi_0) = m_1, \qquad m_2(\varphi_0) = m_2 + \epsilon^2, \qquad m_3(\varphi_0) = m_3 + S\cdot\epsilon\varphi\epsilon.$$

Hence if

$$T\epsilon = 0, \qquad m_2(\varphi) = m_2(\varphi_0).$$

If

$$S\epsilon\varphi\epsilon = 0, \qquad m_3(\varphi) = m_3(\varphi_0).$$

8. Show that

$$m_1[V\epsilon()] = 0, \qquad m_2[V\epsilon()] = T\epsilon^2, \qquad m_3[V\epsilon()] = 0.$$

9. Show that

$$\epsilon(\chi) = -\epsilon, \qquad \epsilon(\chi) = -\varphi\epsilon, \qquad \epsilon(\varphi^{-1}) = -\frac{1}{m_3}\varphi\epsilon.$$

10. Show that

$$\psi(\varphi_0) = \psi_0 + \epsilon S\epsilon().$$

11. If $\theta = V\cdot\beta\varphi()$,

$$m_1(\theta) = 2S\beta\epsilon, \qquad m_2(\theta) = -S\beta\varphi\beta, \qquad m_3(\theta) = 0,$$
$$\epsilon(\theta) = \tfrac{1}{2}\chi(\varphi)\beta, \qquad \psi(\theta) = -\psi\beta_1 S\beta().$$

12. If $\varphi = V\cdot\alpha()$,

$$\varphi^{2n} = \alpha^{2n-2}(\alpha^2 - \alpha S\alpha),$$
$$\varphi^{2n+1} = \alpha^{2n}V\alpha().$$

13. For any two operators φ, θ,

$$2\epsilon(\varphi\theta) = 2\epsilon(\varphi_0\theta_0) + \chi(\varphi)\epsilon(\theta) + \chi(\theta)\epsilon(\varphi) + V\cdot\epsilon(\varphi)\epsilon(\theta).$$

In particular

$$\epsilon(\varphi^2) = \chi(\epsilon),$$
$$\epsilon(\varphi\theta\varphi') = \psi'(\varphi)\epsilon(\theta).$$

14. An operator φ is a *similitude* when for every unit vector α, $T\varphi\alpha = c$, a constant.

Show that the necessary and sufficient condition is

$$\varphi'\varphi = c^2.$$

Any linear transformation which preserves all angles is a similitude.

15. If $\varphi = \alpha Si + \beta Sj + \gamma Sk$, then $\varphi' = iS\alpha + jS\beta + kS\gamma$, and $\varphi\varphi' = -\alpha S\alpha - \beta S\beta - \gamma S\gamma$,

$m_1(\varphi\varphi') = T^2\alpha + T^2\beta + T^2\gamma, \quad m_2(\varphi\varphi') = T^2V\alpha\beta + T^2V\beta\gamma + T^2V\gamma\alpha,$

$$m_3(\varphi\varphi') = -S^2\alpha\beta\gamma.$$

13. **Derivative Dyadic.** There is a dyadic related to a variable vector field of great importance which we will study next. It is called the *derivative dyadic,* since it is somewhat of the nature of a derivative, as well as of the nature of a dyadic. This linear vector function for the field of σ will be indicated by D_σ and defined by the equation

$$D_\sigma = -S()\nabla\cdot\sigma.$$

It is evident at once that if we operate upon $d\rho$, we arrive at $d\sigma$. This function is, therefore, the operator which enables us to convert the various infinitesimal displacements in the field into the corresponding infinitesimal changes in the field itself.

The expression

$$Sd\rho D_\sigma d\rho = Cdt^2,$$

where C is a constant and dt a constant differential, represents an infinitesimal quadric surface, the normals at the ends of the infinitesimal vectors $d\rho$ being $D_\sigma d\rho$.

Let us consider now the field of σ, containing the congruence of vector lines of σ. Consider a small volume given by $\delta\rho$ at the point whose vector is ρ, and let us sup-

pose it has been moved to a neighboring position given by the vector lines of the congruence, that is, ρ becomes $\rho + \sigma dt$. Then $\rho + \delta\rho$ becomes

$$\rho + \delta\rho + dt(\sigma + D_\sigma \delta\rho),$$

that is to say, $\delta\rho$ has become

$$(1 + D\sigma dt)\delta\rho.$$

Hence any area $V\delta_1\rho\delta_2\rho$ becomes, to terms of the first order only,

$$V\delta_1\rho\delta_2\rho + dt(V\delta_1\rho D_\sigma\delta_2\rho + VD_\sigma\delta_1\rho\delta_2\rho).$$

The rate of change with regard to t of the vector area $V\delta_1\rho\delta_2\rho$ is therefore

$$\chi\ (D_\sigma)V\delta_1\rho\delta_2\rho.$$

Likewise, the infinitesimal volume $S\delta_1\rho\delta_2\rho\delta_3\rho$ is transformed into the volume

$$S\delta_1\rho\delta_2\rho\delta_3\rho + dt(S\delta_1\rho\delta_2\rho D_\sigma\delta_3\rho + S\delta_1\rho D_\sigma\delta_2\rho\delta_3\rho + SD_\sigma\delta_1\rho\delta_2\rho\delta_3\rho).$$

The rate of increase of the volume is, therefore, $m_1 S\delta_1\rho\delta_2\rho\delta_3\rho$. In other words if we displace any portion of the space of the medium so that its points travel infinitesimal distances along the lines of the congruence of σ, by amounts proportional to the intensity of the field at the various points, then the change in any infinitesimal line in the portion of space moved is given by $dtD_\sigma\delta\rho$, the change in any infinitesimal area is given by $\chi'(D_\sigma)dt\cdot$Area, and the change in an infinitesimal volume is $m_1 dt$ times the volume.

In case σ defines a velocity field the changes mentioned will actually take place. We have here evidently a most important operator for the study of hydrodynamics. If σdt is the field of an infinitesimal strain, then $D_\sigma\delta\rho$ is the

displacement of the point at $\delta\rho$. Evidently the operator plays an important part in the theory of strain, and consequently of stress. Further, (we shall not stop to prove the result as we do not develop it) for any vector σ a function of ρ we have an expansion analogous to Taylor's theorem, in the series

$$\sigma(\rho_0 + h\alpha) = \sigma(\rho_0) + hD_\sigma\alpha + \frac{h^2}{2}(-S\alpha\nabla)D_\sigma\alpha + \frac{h^3}{6}(S\alpha\nabla)^2 D_\sigma\alpha + \cdots.$$

This formula is the basis of the study of the singularities of the congruence. For if $\sigma(\rho_0) = 0$, then the formula will start with the second term, and the character of the congruence will depend upon the roots of D_σ. In brief the results of the investigation of Poincaré referred to above (p. 38) show that if none of the roots is zero, we have the cases:

1. Roots real and same sign, the singularity is a *node*.
2. Roots real but not all of the same sign, a *faux*.
3. One real root of same sign as real part of other two, a *focus*.
4. One real root of sign opposite the real part of others, a *faux-focus*.
5. One real root, other two pure imaginaries, a *center*.

If one or more roots vanish, we have special cases to consider.

The invariants of D_σ are easily found, and are

$$m_1 = -S\nabla\sigma, \quad \epsilon = \tfrac{1}{2}V\nabla\sigma, \quad m_2 = -\tfrac{1}{2}SV\nabla_1\nabla_2V\sigma_1\sigma_2,$$
$$D_\sigma\epsilon = \tfrac{1}{6}V\cdot V\nabla_1\nabla_2V\sigma_1\sigma_2, \quad m_3 = \tfrac{1}{6}S\nabla_1\nabla_2\nabla_3S\sigma_1\sigma_2\sigma_3.$$

After differentiation, the subscripts are all removed. The related functions are

$$D_\sigma' = -\nabla S\sigma(), \quad \chi = -V\nabla V\sigma(), \quad \chi' = -V\cdot V()\nabla\cdot\sigma,$$
$$\psi = -\tfrac{1}{2}V\nabla_1\nabla_2 S\sigma_1\sigma_2(), \qquad \psi' = -\tfrac{1}{2}S()\nabla_1\nabla_2\cdot V\sigma_1\sigma_2.$$

In a strain σ the dilatation is m_1, the density of rotation (*spin*) is ϵ, and in other cases we can interpret m_1 and ϵ in terms of the convergence and the curl of the field. In case σ is a field of magnetic induction due to extraneous causes, and α is the unit normal of an infinitesimal circuit of electricity, then $\chi'\alpha$ is the negative of the force density per unit current on the circuit. In any case we might call $-\chi' V\delta_1\rho\delta_2\rho$ the force density per unit circuit. Since χ' is not usually self-transverse, the force on circuit α has a component in the direction β different from the component in the direction α of the force on circuit β.

Recurring to Stokes' and Green's theorems we see that

$$\oint d\rho\sigma = \int\int - V\nabla V\delta_1\rho\delta_2\rho\cdot\sigma$$
$$= 2\int\int S\delta_1\rho\delta_2\rho\epsilon - \int\int\chi' V\delta_1\rho\delta_2\rho.$$

It is clear that the circulation in the field of σ is always zero unless for some points inside the circuit ϵ is not zero. The torque of the field on the circuit vanishes for any normal which is a zero axis of χ'. From these it is clear that, if we have a linear function $\varphi d\rho$, in order that it be an exact differential $d\sigma$ we must have the necessary and sufficient conditions

$$V\nabla\varphi() = 0.$$

For if $\oint\varphi d\rho = 0$, then $\varphi VU\nu\nabla = 0$ for all $U\nu$, whence the condition. The converse is easy.

The invariant m_3 in the case of the points at which $\sigma = 0$ will be sometimes positive, sometimes negative. A theorem given originally by Kronecker enables us to find what the excess of the number of roots at which m_3 is positive over the number of roots at which m_3 is negative is.* We set

* Picard, Traité d'Analyse, Vol. I, p. 139.

$$\tau = \psi\sigma / T\sigma^3 \qquad \text{and} \qquad I = -\frac{1}{4\pi} \oint\oint S d\nu \tau;$$

then the integral will vanish for any space containing no roots, and will be the excess in question for any other space. We could sometimes use this theorem to determine the number of singularities in a region of space and something about their character. It is evident that $S\nabla\tau = 0$.

The operator $(D_\sigma)_0 = \frac{1}{2}(D_\sigma + D_\sigma')$ is called the *deformation* of the field, and the operator $V\epsilon()$ the *rotation* of the field.

In case σ is a unit vector everywhere, then $D_\sigma'\sigma = 0$, and since the transverse has a zero root, D_σ itself must have a zero root. There is one direction then for which $D_\sigma\alpha = 0$. The vector lines given by $V\alpha d\rho = 0$ are the isogons of the field. In case there are two zero roots the isogons are any lines on certain isogon surfaces.

EXERCISES

1. Study the fields given by

$$\sigma = -\rho, \qquad \sigma = U\rho/\rho^2, \qquad \sigma = V\alpha\rho, \qquad \sigma = \alpha S\beta\rho, \qquad \sigma = V\alpha\rho/\rho^3.$$

2. Show that if σ is a function of ρ,

$$\begin{aligned} \sigma + d\sigma &= -\nabla_0[S\rho_0\sigma - \tfrac{1}{2}S\rho_0\varphi\rho_0] - \tfrac{1}{2}V\rho_0 V\nabla\sigma \\ &= V\nabla_0[\tfrac{1}{2}V\sigma\rho_0 - \tfrac{1}{3}V\rho_0\varphi\rho_0] - \tfrac{1}{3}S\nabla\sigma, \end{aligned}$$

where ∇_0 operates only on ρ_0, and $\varphi = -\sigma S\nabla()$. The first form expresses $\sigma + d\sigma$ as a gradient and a term dependent on the curl of σ, the second as a curl and a term dependent on the convergence of σ. ρ_0 is an infinitesimal vector.

3. If $\sigma = V\nabla\tau$, $D_\sigma = D_\sigma'$.

14. **Dyadic Field.** If φ is a linear vector operator dependent upon ρ, we say that φ defines a *dyadic field.* For every point in space there will be a value of φ. Since there is always one root at least for φ which is real, with an invariant line, there will be for every point in space a direction

and a numerical value of the root which gives the real invariant direction and root. These will define a congruence of lines and a numerical value along the lines. In case the other axes are also real, and the roots are distinct or practically distinct, there will be two other related congruences. The study of the structure of a dyadic field from this point of view will not be entered into here, but it is evidently of considerable importance.

EXERCISES

1. If $\varphi = u()$, then the *gradient* of the field is ∇u. The *vorticity* of the field is $V\nabla\varphi() = V\nabla u()$. The gradient in any case is $\varphi\nabla$, a vector.

2. If $\varphi = V\sigma()$, the gradient is $-V\nabla\sigma$, the vorticity is

$$()S\nabla\sigma + D_\sigma = -\chi(D_\sigma).$$

3. If $\varphi = \sigma S\tau()$, the gradient is $\sigma S\nabla\tau - D_\tau\sigma$, the vorticity is $V\nabla\sigma S\tau() + V\sigma D_\gamma()$. The gradient of the transverse field is $\tau S\nabla\sigma - D_\sigma\tau$, the vorticity $V\nabla\tau S\sigma() + V\tau D_\sigma()$.

4. If $\varphi = V\sigma\theta()$, the gradient is $-V\theta(\nabla)\sigma + V\sigma\theta'\nabla'$, the vorticity is

$$S\nabla\sigma\cdot\theta() + S\sigma\nabla'\cdot\theta'() - \sigma S\theta_1'(\nabla_1)() + D_\sigma\,\theta().$$

For the transverse field we have

$$\begin{aligned}&\text{the gradient is } -\theta'V\nabla'\sigma - \theta V\nabla\sigma,\\ &\text{the vorticity } V\nabla'\theta'V\sigma() + V\nabla'\theta V\sigma'().\end{aligned}$$

5. If $\varphi = D_\sigma$ the gradient of the field is $-\nabla^2\sigma$, the concentration of σ, and the vorticity is $D_{V\nabla\sigma}$. The gradient of the transverse field is $-\nabla S\nabla\sigma$, while the vorticity is zero.

6. If $\varphi = V\nabla\theta()$, the gradient is $V\nabla\theta\nabla$, where both ∇'s act on θ, and the vorticity is $\nabla^2\theta() - \nabla S\nabla\theta()$.

7. If $\varphi = D_{\theta(\sigma)}$, the gradient is $-\nabla^2\theta\sigma$, the vorticity is $D_{V\nabla\theta\sigma}$.

8. If $\varphi = \psi_\theta$, the gradient is $2\epsilon(\theta V\bar{\nabla}\theta)$.

9. For any φ

$$\begin{aligned}\nabla m_1 &= \varphi\nabla + 2\,\epsilon\,(V\nabla\varphi\,()\,),\\ \nabla m_2 &= 2\,\epsilon\,[\varphi V\nabla\varphi' + V\nabla\psi'\,()\,],\\ \nabla m_3 &= 2\,\epsilon\,[V\,(\nabla_1\,\psi_1')\,\varphi'\\ &\qquad -\psi\,\varphi_1'\,\nabla_1\,].\end{aligned}$$

15. **The Differentiator.** We define the operator $-S()\nabla$ as the differentiator, and indicate it by D. It may be used upon quaternions, vectors, scalars, or dyadics.

As examples we have, $\bar{D}$ being the transverse

$$D_{V\sigma\tau} = V\sigma\bar{D}_\tau() - V\tau\bar{D}_\sigma(), \qquad D_{S\sigma\tau} = S()D_\sigma\tau + S()D_\tau\sigma,$$
$$D_{V\sigma\alpha} = -V\alpha\bar{D}_\sigma(), \qquad D_{m_1}(\varphi) = m_1(D_\varphi),$$
$$D_{\epsilon(\varphi)} = \epsilon(D_\varphi), \qquad D_\varphi = -S()\nabla\cdot\varphi().$$

16. **Change of Variable.** Let F be a function of ρ, and ρ a function of three parameters u, v, w. Let

$$\triangle = \alpha\partial/\partial u + \beta\partial/\partial v + \gamma\partial/\partial w,$$

where α, β, γ form a right-handed system of unit vectors. Then we have the following formulae to pass from expressions in terms of ρ to differential expressions in terms of the parameters.

$$\triangle F = -\triangle_1 S\rho_1\nabla F,$$
$$V\triangle'\triangle'' = \tfrac{1}{2}V\triangle_1'\triangle_2''SV\rho_1\rho_2V\nabla'\nabla'',$$
$$S\triangle'\triangle''\triangle''' = -\tfrac{1}{6}S\triangle_1'\triangle_2''\triangle_3'''S\rho_1\rho_2\rho_3S\nabla'\nabla'\nabla'''.$$

As instances

$$-S\nabla\sigma = \triangle'S\rho'\nabla''\cdot\sigma'',$$
$$V\triangle\sigma = S\rho'\nabla''\cdot V\sigma''\triangle'.$$

Notations

Dyadic products

$\phi(a)$, $\phi'(a)$, $\phi Va(\ )$, $Va\phi(\ )$, Hamilton, Tait, Joly, Shaw.
$\phi\cdot a$, $a\cdot\phi$, $\phi\times a$, $a\times\phi$, Gibbs, Wilson, Jaumann, Jung.

Reciprocal dyadic

ϕ^{-1}, Hamilton, Tait, Joly, Gibbs, Wilson, Burali-Forti, Marcolongo, Shaw.
q^{-1}, Timerding.
$|b|^{-1}$, Élie.

The adjunct dyadic

$\psi = m\phi'^{-1}$, Hamilton, Tait, Joly, Shaw.
$(\phi_c)_2$, Gibbs, Wilson, Macfarlane.
$R(\alpha)$, Burali-Forti, Marcolongo.
$\chi(\phi, \phi)$, Shaw.
$D\phi^{-1}$, Jaumann, Jung.

The transverse or conjugate dyadic

ϕ', Hamilton, Tait, Joly.
$\check{\phi}$, Taber, Shaw.
ϕ_c, Gibbs, Wilson, Jaumann, Jung, Macfarlane.
$K(\alpha)$, Burali-Forti, Marcolongo.
$\backslash\, b\, /$, Élie.

The planar dyadic

$\chi = m_1 - \phi'$, Hamilton, Tait, Joly.
$\phi_s I - \phi_c$, Gibbs, Wilson.
$- \phi_c{}^r$, Jaumann, Jung.
$CK(\alpha)$, Burali-Forti, Marcolongo.
$\chi(\check{\phi})$, Shaw.

Self-transverse or symmetric part of dyadic

ϕ_0, Hamilton, Tait, Shaw.
Φ, Joly.
ϕ', Gibbs, Wilson.
$[\phi]$, Jaumann, Jung.
$D(\alpha)$, Burali-Forti, Marcolongo.
$\backslash\; b\; /$, Élie.
$\backslash\; b^0\; /$, Élie. In this case expressed in terms of the axes.

Skew part of dyadic

$\frac{1}{2}(\phi - \phi') = V\cdot\epsilon(\;)$, Hamilton, Tait, Joly, Shaw.
ϕ'', Gibbs, Wilson.
Π, Jaumann, Jung.
$V\alpha\wedge$, Burali-Forti, Marcolongo.

$\setminus \bar{b} /$, Élie.
Sin ϕ, Macfarlane.

Mixed functions of dyadic

$\chi(\phi, \theta)$, Shaw.
$\frac{1}{2}\phi_{\times}^{\times} \theta$, Gibbs, Wilson.
$R(\phi, \theta)$, Burali-Forti, Marcolongo.

Vector of dyadic

ϵ, Hamilton, Tait, Joly.
$\phi_{\times}$, Gibbs, Wilson.
$\phi_r{}^s$, $-\ \phi_r{}^r$, Jaumann, Jung.
$V\alpha$, Burali-Forti, Marcolongo.
$\overline{E}$, Carvallo.
$R = T\epsilon$, Êlie.
$\epsilon(\phi)$, Shaw.

Negative vector of adjunct dyadic

$\phi\epsilon$, Hamilton, Tait, Joly.
$\phi \cdot \phi_{\times}$, Gibbs, Wilson.
$\phi \cdot \phi_r{}^s$, Jaumann, Jung.
$\alpha V\alpha$, Burali-Forti, Marcolongo.
$\epsilon\chi(\phi, \phi)$, Shaw.

Square of pure strain factor of dyadic

$\phi\phi'$, Hamilton, Tait, Joly.
$\phi\phi_c$, Gibbs, Wilson.
$\{\phi\}^2$, Jaumann, Jung.
$\alpha K\alpha$, Burali-Forti, Marcolongo.
$[b]$, Élie.
$\phi\phi'$, Shaw.

Dyadic function of negative vector of adjunct

$\phi^2\epsilon$, Hamilton, Tait, Joly, Shaw.
$\phi^2 \cdot \phi_{\times}$, Wilson, Gibbs.

$\phi^2 \cdot \phi_r{}^s$, Jaumann, Jung.
$\alpha^2 V\alpha$, Burali-Forti, Marcolongo.
K_2, Élie.

Scalar invariants of dyadic. Coefficients of characteristic equation

m'', m', m, Hamilton, Tait, Joly, Carvallo.
I_1, I_2, I_3, Burali-Forti, Marcolongo, Élie.
F, G, H, Timerding.
ϕ_s, $(\phi_2)_s$, ϕ_3, Gibbs, Wilson.
m_1, m_2, m_3, Shaw.

$$\left.\begin{array}{l}\phi_s, \\ \phi_s{}^s, \\ -\frac{1}{2}\phi_s{}^r,\end{array}\right\} \cdots \phi_3, \text{ Jaumann, Jung.}$$

$\cos\phi \cdots \phi_3$, Macfarlane.

Other scalar invariants

$m_1(\phi_0{}^2)$, $m_1(\phi\phi')$, $2(m_1{}^2 - m_2)$, $m_1(\phi\theta')$, $m_1[\chi(\phi, \theta), \beta]$, Shaw.
$[\phi^s]^2{}_s$, $\{\phi\}_s{}^2$, $[\phi^r]_s{}^2$, $\cdots$, Jaumann, Jung.
$\cdots, \cdots, \cdots, \phi : \theta$, $\phi \overset{\times}{\times} \theta : \beta$, Gibbs, Wilson.
Élie uses K_1 for $S\epsilon\phi\epsilon$.

Notations for Derivatives of Dyadic

In these ∇ operates on ϕ unless the subscript $_n$ indicates otherwise.

Gradient of dyadic

$\nabla\phi$, Tait, Joly, Shaw.

Dyadic of gradient. Specific force of field

$\phi\nabla$, Tait, Joly, Shaw.
grad α, Burali-Forti, Marcolongo.
$\frac{d \cdot \phi}{dr}$, Fischer.

Transverse dyadic of gradient

$\phi'\nabla$, Tait, Joly.

grad $K\alpha$, Burali-Forti, Marcolongo.

$\dfrac{d\cdot\phi_c}{dr}$, Fischer.

$\nabla\cdot\phi$, Jaumann, Jung.

Divergence of dyadic

$-S\nabla\phi(\ )$, Tait, Joly, Shaw.

$\times$ grad $K\alpha$, Burali-Forti, Marcolongo.

Vortex of dyadic

$V\nabla\phi(\ )$, Tait, Joly, Shaw.

Rot α, Burali-Forti.

$\nabla\times\phi$, Jaumann, Jung.

Directional derivatives of dyadic

$-S(\ )\nabla\cdot\phi$. $Sa^{-1}\nabla\cdot\phi a$. $Sa^{-1}\nabla\cdot\phi Va(\ )$, Tait, Joly, Shaw.

$S(\alpha, (\ ))$, Burali-Forti.

$\dfrac{\partial\cdot\phi}{\partial a}$, $\dfrac{\partial\times\phi}{\partial a}$, Fischer.

$\left(\dfrac{d\alpha}{dP}(\ )\right)(\ )$, Burali-Forti, Marcolongo.

Gradient of bilinear function

$\mu_n(\nabla_n, a)$, Tait, Joly, Shaw.

$\phi(\mu)a$, Burali-Forti.

Bilinear gradient function

$\mu(\nabla_n, u_n)$, Tait, Joly, Shaw.

$\psi(\mu, u)$, Burali-Forti.

Planar derivative of dyadic

$\phi_n V\nabla_n(\ )$, Tait, Joly, Shaw.

$\times\dfrac{\phi}{dr}$, Fischer.

CHAPTER X

DEFORMABLE BODIES

Strain

1. When a body has its points displaced so that if the vector to a point P is ρ, we must express the vector to the new position of P, say P', by some function of ρ, $\varphi\rho$, then we say that the body has been strained. We do not at first need to consider the path of transition of P to P'. If φ is a linear vector function, then we say that the strain is a linear homogeneous strain. We have to put a few restrictions upon the generality of φ, since not every linear vector function can represent a strain. In the first place we notice that solid angles must not be turned into their symmetric angles, so that $S\varphi\lambda\varphi\mu\varphi\nu/S\lambda\mu\nu$ must be positive, that is, m_3 is positive. Hence φ must have either one or three positive real roots. The corresponding invariant lines are, therefore, not reversed in direction.

2. When φ is self-conjugate there are three real roots and three directions which form a trirectangular system. The strain in this case is called a *pure* strain. Any linear vector function can be written in the form

$$\varphi = \sqrt{(\varphi\varphi')}\cdot q^{-1}()q = p^{-1}\sqrt{(\varphi'\varphi)}()\cdot p,$$

where

$$q^{-1}()q = (\varphi\varphi')^{-1/2}\varphi.$$

The function $\varphi\varphi'$ is self-conjugate and, therefore, has three real roots and its invariant lines perpendicular. If we set $\pi = \sqrt{(\varphi\varphi')}$, then $\pi^2 = \varphi\varphi'$. Let the cubic in $\varphi\varphi'$ be $G^3 - M_1G^2 + M_2G - M_3 = 0$. Then from the values given in Chapter IX, p. 237, for the coefficients of φ^2

in terms of those of φ we have (the coefficients of the cubic in π being p_1, p_2, p_3)

$$M_1 = p_1^2 - 2p_2, \qquad M_2 = p_2^2 - 2p_1p_3, \qquad M_3 = p_3^2,$$

whence we have

$$p_1^4 - 2(M_1 + 8M_3)p_1^2 - 16M_2M_3p_1 + M_1^2 - 4M_2^2M_3 = 0.$$

Thence we have p_1, p_2, and p_3.

Now if the invariant lines of $\varphi\varphi'$ are the trirectangular unit vectors α, β, γ, we may collect the terms of φ in the form

$$\varphi = a\alpha S\alpha'() + b\beta S\beta'() + c\gamma S\gamma'(),$$

where a, b, c are the roots of $\sqrt{\varphi\varphi'} = \pi$ and α', β', γ' are to be determined. Hence $\varphi' = a\alpha' S\alpha() + \cdots$ and

$$-\varphi'\varphi = a^2\alpha' S\alpha'() + b^2\beta' S\beta'() + c^2\gamma' S\gamma'().$$

But also

$$\varphi\varphi' = -a^2\alpha S\alpha - b_2\beta S\beta - c^2\gamma S\gamma,$$

since α, β, γ are axes of $\varphi\varphi'$, and a^2, b^2, c^2 are roots. Now we have

$$\varphi'\alpha = -a\alpha', \qquad \varphi'\beta = -b\beta', \qquad \varphi'\gamma = -c\gamma',$$

hence

$$\varphi\varphi'\alpha = a^2\alpha = -a^2\alpha S\alpha'\alpha' - ab\beta S\alpha'\beta' - ac\gamma S\alpha'\gamma'.$$

Thus we have $\alpha'^2 = -1$, $S\alpha'\beta' = 0 = S\alpha'\gamma'$, and similar equations, so that α', β', γ' are unit vectors forming a trirectangular system, and indeed are the invariant lines of $\varphi'\varphi$. We may now write at once

$$\pi = -a\alpha S\alpha - b\beta S\beta - c\gamma S\gamma,$$
$$q^{-1}()q = -\alpha S\alpha' - \beta S\beta' - \gamma S\gamma'.$$

This operator obviously rotates the system α', β', γ' into

the system α, β, γ, as a rigid body. That the function is orthogonal is obvious at a glance, since if we multiply it by its conjugate we have for the product

$$-\alpha S\alpha - \beta S\beta - \gamma S\gamma = 1().$$

Reducing it to the standard form of example five, Chapter IX, p. 236, we find that the axis is $UV(\alpha\alpha' + \beta\beta' + \gamma\gamma')$ and the sine of the angle of rotation $\frac{1}{2}TV(\alpha\alpha' + \beta\beta' + \gamma\gamma')$.

EXAMPLES

(1). Let $\varphi = V\epsilon()$. Then

$$\varphi' = -V\epsilon(), \qquad \varphi\varphi' = -V\epsilon V\epsilon() = \epsilon S\epsilon() - \epsilon^2.$$

The axes are ϵ for the root 0, and any two vectors α, β perpendicular to ϵ, and these must be taken so that $\alpha\beta = U\epsilon$, the roots that are equal being $T^2\epsilon$. We may therefore write

$$\varphi = T\epsilon\alpha S\beta - T\epsilon\beta S\alpha = V\cdot\epsilon(),$$

which was obvious anyhow. Hence we have for $q^{-1}()q$ the operator

$$\alpha S\beta - \beta S\alpha = V(V\alpha\beta)(),$$

and this is a rotation of 90° about $V\alpha\beta = V\epsilon$ of 90°. The effect of

$$\pi = T\epsilon(-\alpha S\alpha - \beta S\beta)$$

is to give the projection of the rotated vector on the plane perpendicular to ϵ, times $T\epsilon$. That is, finally, $V\epsilon()$ rotates ρ about ϵ as an axis through 90° and annuls the component of the new vector which is parallel to ϵ.

(2). Consider the operator $g - \alpha S\beta()$ where α, β are any vectors. It is to be noticed that we must select of all the square roots of $\varphi\varphi'$ that one which has its roots all positive. It is obvious that $p = q$.

3. The strain converts the sphere $T\rho = r$ into the ellipsoid $T\varphi^{-1}\rho = r$, or

$$S\rho\varphi'^{-1}\varphi^{-1}\rho = -r^2.$$

This is called the strain ellipsoid. Its axes are in the directions of the perpendicular system of $\varphi\varphi' = \pi^2$. The ellipsoid $S\rho\varphi'\varphi\rho = -r^2$ is converted into the sphere $T\rho = r$. This is the reciprocal strain ellipsoid. Its axes are in the directions of the principal axes of the strain. The extensions of lines drawn in these directions in the state before the strain are stationary, and one of them is thus the maximum, one the minimum extension.

4. A shear is represented by

$$\varphi\rho = \rho - \beta S\alpha\rho,$$

where $S\alpha\beta = 0$. The displacement is parallel to the vector β and proportional to its distance from the plane $S\alpha\rho = 0$. There is no change in volume since $m_3 = 1$.

If there is a uniform dilatation and a shear the function is

$$\varphi\rho = g\rho - \beta S\alpha\rho.$$

The change in volume is now g^3. The equation is easily seen to be

$$(\varphi - g)^2 = 0.$$

This is the necessary and sufficient condition of a dilatation and a shear, but this equation alone will not give the axes and the shear plane, of course.

5. The function $\varphi\rho = gq\rho q^{-1} - q\beta q^{-1}S\alpha\rho$ is a form into which the most general strain can be put which is due to shifting in a fixed direction, $U\beta$, planes parallel to the fixed plane $S\alpha\rho = 0$ by an amount proportional to the perpendicular distance from the fixed plane, then altering all lines in the ratio g, and superposing a rotation. This is

any strain. We simply have to put $\varphi'\varphi$ into the form

$$\varphi'\varphi = b^2 + \lambda S\mu + \mu S\lambda,$$

where

$$S\lambda\mu = \tfrac{1}{2}(a^2 + c^2 - 2b^2), \qquad T\lambda\mu = \tfrac{1}{2}(a^2 - c^2),$$

and then we take

$$g = b, \qquad \alpha = -\lambda, \qquad b\beta = \mu - \tfrac{1}{2}\lambda^{-1}(a - c)^2.$$

The rotation is determined as before.

6. All the lines in the original body that are lengthened in the same ratio, say g, are parallel to the edges of the cone $T\varphi U\rho = g$ or $SU\rho(\varphi'\varphi - g^2)U\rho = 0$, or in terms of λ, μ, $2S\lambda U\rho S\mu U\rho = b^2 - g^2$, $\sin u \cdot \sin v = (b^2 - g^2)/(a^2 - c^2)$, where u and v are the angles the line makes with the cyclic planes of the cone $S\varphi\rho\varphi\rho = -T^2\rho$.

7. The displacement of the extremity of ρ is

$$\delta = \sigma - \rho = (\varphi - 1)\rho,$$

which can be resolved along ρ and perpendicular to ρ into the parts

$$\rho(S\rho^{-1}\varphi\rho - 1) + \rho V\rho^{-1}\varphi\rho.$$

The coefficient of ρ in the first term is called the elongation. It is numerically the reciprocal of the square of the radius of the elongation quadric:

$$S\rho(\varphi_0 - 1)\rho = -1,$$

the radius being parallel to ρ.

The other component may be written $V\epsilon\rho + V\varphi_0\rho\rho^{-1}\cdot\rho$, where ϵ is the invariant vector of φ, the spin-vector.

8. If now the strain is not homogeneous, we must consider it in its infinitesimal character. In this case we have again the formula $d\sigma = -Sd\rho\nabla\cdot\sigma = \varphi d\rho$, where σ is now the displacement of P, whose vector is ρ, and $\sigma + d\sigma$ that of

$\rho + d\rho$, provided that we can neglect terms of the second order. If these have to be considered,

$$\begin{aligned} d\sigma &= -\, Sd\rho\nabla\cdot\sigma + \tfrac{1}{2}(Sd\rho\nabla)\nabla^2\sigma \\ &= \varphi d\rho - \tfrac{1}{2}Sd\rho\nabla\cdot\varphi d\rho. \end{aligned}$$

We may analyze the strain in the case of first order into

$$\varphi = \varphi_0 + V\epsilon().$$

Since now $\epsilon = \frac{1}{2}V\nabla\sigma$, if $\epsilon = 0$, it follows that $\sigma = \nabla P$ and there is a displacement potential and

$$\varphi = -\,\nabla S\nabla P().$$

The strain is in this case a pure strain. If ϵ is not zero, there is rotation, about ϵ as an axis, of amount $T\epsilon$. In any case the function φ_0 determines the changes of length of all lines in the body, the extension e of the short line in the direction $U\rho$ being

$$-\,SU\rho\varphi_0U\rho.$$

The six coefficients of φ_0, of form $-\,S\alpha\varphi_0\beta$, where α, β are any two of the three trirectangular vectors α, β, γ, are called the components of strain. Three are extensions and three are shears, an unsymmetrical division.

9. In the case of small strains the volume increase is $-\,S\nabla\sigma$, and this is called the cubic dilatation. If it vanishes, the strain takes place with no change of volume, that is, with no change of density. A strain of this character is called a transversal strain. There is a vector potential from which σ can be derived by the formula

$$\sigma = V\nabla\tau, \qquad S\nabla\tau = 0.$$

There is no scalar potential since we do not generally have also $V\nabla\sigma = 0$. Indeed we have

$$2\epsilon = V\nabla\sigma = V\nabla V\nabla\tau = \nabla^2\tau - \nabla S\nabla\tau = \nabla^2\tau.$$

This would give us the integral

$$\tfrac{1}{2}\tau = \tfrac{1}{4}\pi \int\int\int \epsilon/r \cdot dv.$$

The integration is over the entire body.

This strain is called transverse because in case we have σ a function of a single projection of ρ, on a given line, say α, so that

$$\sigma = \alpha f_1 \cdot x + \beta f_2 \cdot x + \gamma f_3 \cdot x,$$
$$S\nabla\sigma = -f_1 = 0, \qquad f_1 = \text{constant},$$

and all points are moved in this direction like those of a rigid body. We may therefore take the constant equal to zero, and $f_1 = 0$, so that

$$S\alpha\sigma = 0.$$

Hence every displacement is perpendicular to the line α.

10. When $V\nabla\sigma = 0$, we call the strain longitudinal; for, giving σ the same expression as in § 9, we see that we have

$$V\nabla\sigma = 0 = \gamma f_2' - \beta f_3', \qquad \text{and} \qquad f_2 = 0 = f_3,$$
$$V\alpha\sigma = 0.$$

Hence we have all the strain parallel to α.

11. In case the cubical dilatation $S\nabla\sigma = 0$, the strain is purely of a shearing character, and if the curl $V\nabla\sigma = 0$, the strain is purely of a dilatational character. Since any vector σ can be separated into a solenoidal and a lamellar part in an infinity of ways, it is always possible to separate the strain into two parts, one of dilatation only, the other of shear only.

If we write $\sigma = \nabla P + V\nabla\tau$, then we can find P and τ in one way from the integrals

$$P = \tfrac{1}{4}\pi \cdot \int\int\int S\sigma'\nabla T\rho^{-1} dv',$$
$$\tau = -\tfrac{1}{4}\pi \cdot \int\int\int V\sigma'\nabla T\rho^{-1} \cdot dv', \qquad \rho = \rho' - \rho_C.$$

The integrations extend throughout the body displaced. This method of resolution is not always successful, and other formulae must be used. (Duhem, Jour. des Math., 1900.)

12. The components are not functionally independent, but are subject to a set of relations due to Saint Venant. These relations are obvious in the quaternion form, equivalent to six scalar equations. The equation is

$$V\cdot\nabla\varphi_0 V\nabla() = 0, \qquad \text{if} \qquad \varphi = S()\nabla\cdot\sigma,$$

where both ∇'s operate on φ_0. The equation is, furthermore, the necessary and sufficient condition that any linear vector function φ can represent a strain. The problem of finding the vector σ when φ is a given linear and vector function of ρ consists in inverting the equation

$\varphi = -S()\nabla\cdot\sigma.$ (Kirchkoff, Mechanik, Vorlesung 27.)

It is evident that if we operate upon $d\rho$, we have

$$\varphi d\rho = d\sigma.$$

Hence the problem reduces to the integration of a set of differential equations of the ordinary type.

EXAMPLES

(1). If $\varphi = V\epsilon()$, we have $\sigma = V\epsilon\rho$. Prove Saint Venant's equations.

(2). If $\varphi = \rho^{-1}V()\rho^{-1}$, then $\sigma = \mathring{U}\rho$. Prove Saint Venant's equations.

13. In general when we do not have small strains, we must modify the preceding theory somewhat. The displacement will change the differential element $d\rho$ into

$$d\rho_1 = d\rho - Sd\rho\nabla\cdot\sigma.$$

The strain is characterized when we know the ratio of the two differential elements and this we may find by squaring

so as to arrive at the tensor

$$(d\rho_1)^2 = Sd\rho[1 - 2\nabla S\sigma + \nabla' S\sigma'\sigma'' S\nabla'']d\rho.$$

The function in the brackets is the general strain function, which we will represent by Φ. It is easily clear that if $\varphi = - S()\nabla \cdot \sigma$ then

$$\Phi = (1 + \varphi)(1 + \varphi') = (1 + \varphi)(1 + \varphi)'.$$

Of course Φ is self-conjugate. Its components $S\alpha\Phi\beta$ are also called components of strain. If φ is infinitesimal, we may substitute $(1 + 2\varphi_0)$ for Φ.

The cubical dilatation is now found by subtracting 1 from

$$Sd_1\rho_1 d_2\rho_1 d_3\rho_1 / Sd_1\rho d_2\rho d_3\rho = m_3(1 + \varphi) = 1 + \triangle.$$

Evidently $(1 + \triangle)^2 = m_3(\Phi)$. The alteration in the angle of two elements is found from

$$- SU(1 + \varphi)\lambda U(1\varphi)\lambda'.$$

If angles are not altered between the infinitesimal elements, the transformation is conformal, or isogonal. In such case

$$S^2\lambda\Phi\lambda' = S^2\lambda\lambda' S\lambda\Phi\lambda S\lambda'\Phi\lambda'.$$

For example, if $\varphi = V\alpha()$,

$$SU(1 + \varphi)\lambda(1 + \varphi)\lambda' = S\lambda\lambda',$$

when $S\alpha\lambda = 0 = S\alpha\lambda'$.

14. This part of the subject leads us into the theory of infinitesimal transformations, and is too extensive to be treated here.

On Discontinuities

15. If the function σ is continuous throughout a body, it may happen that its convergence or its curl may be discontinuous. The consideration of such discontinuities is

usually given at length in a discussion of the potential functions. Here we need only the elements of the theory. We make use of the following general theorem from analysis.

LEMMA. *If a function is continuous on one side of a surface for all points not actually on the surface in question, and if, as we approach the surface by each and every path leading up to a point P, the gradient of the function, or its directional derivatives approach one and the same limit for all the paths; then the differential of this function along a path lying on the surface is also given by the usual formula,*

$$- Sd\rho\nabla \cdot q = dq, \qquad d\rho \text{ being on the surface.}$$

[Hadamard, Leçons sur la propagation des ondes, etc., p. 84, Painlevé, Ann. École Normale, 1887, Part 1, ch. 2, no. 2.]

In the case of a vector σ which has the same value on each side of a surface, which is the value on the surface, and is the limiting value as the surface is approached, at all points of the surface, we have on one side of the surface

$$d\sigma = - Sd\rho\nabla \cdot \sigma = \varphi_1 d\rho.$$

On the opposite side

$$d\sigma = - Sd\rho\nabla \cdot \sigma = \varphi_2 d\rho.$$

If now these two do not agree, but there is a discontinuity in φ, so that $\varphi_2 - \varphi_1$ is finite as the two paths are made to approach the surface, then designating the fluctuation or saltus of a function by the notation [], we have in the limit

$$[d\sigma] = (\varphi_2 - \varphi_1)d\rho = [\varphi]d\rho.$$

But since σ does not vary abruptly, $[d\sigma]$ along the surface is zero, hence for $d\rho$ on the surface

$$[\varphi]d\rho = 0,$$

and therefore

$$[\varphi] = -\mu S\nu,$$

where ν is the unit normal, μ a given vector. That is to say, we have for the transition of the surface

$$[S()\nabla\cdot\sigma] = \mu S\nu.$$

Whence

$$[S\nabla\sigma] = S\nu\mu,$$
$$[V\nabla\sigma] = V\nu\mu.$$

These are conditions of compatibility of the surface of discontinuities and the discontinuity; or identical conditions, under which the discontinuities can actually have the surface for their distribution.

16. If $S\mu\nu = 0$, then $[S\nabla\sigma] = 0$, and the cubic dilatation is continuous.

Since $S\nu\nu\mu = 0 = S\nu[V\nabla\sigma] = [S\nu\nabla\sigma]$, the normal component of the curl of σ is continuous, and the discontinuity is confined to the tangential component. Likewise

$$S\mu\nu\mu = 0 = [S\mu\nabla\sigma],$$

and the component along μ is continuous. Hence $V\nabla\sigma$ can be discontinuous only normal to the plane of μ, ν.

17. In case σ itself is discontinuous, the normal component of σ as it passes the surface of discontinuity cannot be discontinuous without tearing the surface in two. Hence the discontinuity is purely tangential. It can be related to the curl of σ as follows.

Consider a line on the surface, of infinitesimal length, and an infinitesimal rectangle normal to the surface, and let the value of σ at the two upper points differ only infinitesimally, as likewise at the two lower points, but the difference at the two right hand points or at the two left hand

points be finite, so that σ has a discontinuity in going through the surface equal to $[\sigma]$. Then

$$\oint S\delta\rho\sigma = \int\int S\kappa(AV\nabla\sigma)$$

around the rectangle, when κ is normal to the rectangle. But the four parts on the left for the four sides give simply

$$S\delta\rho[\sigma],$$

where $\delta\rho$ is a horizontal side and equal to $V\cdot\kappa\nu T\delta\rho$. Hence we have for every κ tangential to the surface

$$S\kappa V\nu[\sigma] = S\kappa \text{ Lim } (AV\nabla\sigma)/T\delta\rho.$$

Dropping all infinitesimals, we have

$$V\nu[\sigma] = \text{Lim } AV\nabla\sigma/T\delta\rho.$$

Tangential discontinuities may therefore be considered to be representable by a limiting value of the curl multiplied by an infinitesimal area, as if the surface of discontinuity were the locus of the axial lines of an infinity of small rotations which enable one space to roll upon the other. The expression $\frac{1}{2}[\sigma]$ is the *strength* of this sheet.

A strain is not irrotational unless such surfaces of discontinuity are absent. But we have shown above that a continuous strain may imply certain surfaces of discontinuity in its derivatives of some order. If $V\nabla\sigma = 0$, everywhere, then $V\nu[\sigma] = 0$, and such discontinuity as exists is parallel to ν.

The derivation above applies to any case, and we may say that if a field is irrotational, any discontinuity it possesses must be normal to the surface of discontinuity.

Integrating in the same way over the surface of a small box, we would have

$$\oint\oint S\nu[\sigma]ds = S\nabla\sigma\cdot v,$$

where v is the infinitesimal volume. But this gives

$$S\nu[\sigma] = vS\nabla\sigma/\text{surface}.$$

If then $S\nabla\sigma = 0$ everywhere, the discontinuity of σ is normal to the normal, that is, it is purely tangential. These theorems will be useful in the study of electro-dynamics.

Kinematics of Displacements

18. In the case of a continuous displacement which takes place in time we have as the vector σ the velocity of a moving particle, and if ρ is the vector from a fixed point to the particle, then $d\rho/dt = \sigma$. It is necessary to distinguish between the velocity of the particle and the local velocity of the stream of particles as they pass a given fixed point in the absolute space which is supposed to be stationary. The latter is designated by $\partial/\partial t$. Thus $\partial\sigma/\partial t$ is the local rate of change of the velocity at a certain point. While $d\sigma/dt$ is the rate of change of the velocity as we follow the particle. It is easy to see that for any quaternion q the actual time rate of change is

$$dq/dt = \partial q/\partial t - S\sigma\nabla\cdot q.$$

We have thus the acceleration

$$d\sigma/dt = \partial\sigma/\partial t - S\sigma\nabla\cdot\sigma = (\partial/\partial t + \varphi)\sigma.$$

If the infinitesimal vector $d\rho$ is considered to be displaced, we have

$$\delta d\rho/dt = - S\delta\rho\nabla\cdot\sigma.$$

Since the rotation is $\frac{1}{2}V\nabla\sigma\, dt$, the angular velocity of turn of the particle to which $d\rho$ is attached is $\frac{1}{2}V\nabla\sigma$. This is the vortex velocity. Likewise the velocity of cubic dilatation is $- S\nabla\sigma$.

The rate of change of an infinitesimal volume dv as it

moves along is

$$- S\nabla\sigma \cdot dv.$$

The equation of continuity is $d(cdv) = 0$, where c is the density, or

$$dc/dt + c(- S\nabla\sigma) = 0.$$

That is, we have for a medium of constant mass

$$dc/dt = cS\nabla\sigma.$$

That is, the density at a moving point has a rate of change per second equal to the density times the convergence of the velocity.

It may also be written easily

$$\partial c/\partial t = S\nabla(c\sigma).$$

This means that at a fixed point the velocity of increase in density is equal to the convergence of the momentum per cubic centimeter.

19. When $V\nabla\sigma = 0$, the motion is irrotational, or dilatational, and we may put $\sigma = \nabla P$, where now P is a velocity-potential, which may be monodromic or polydromic. When $S\nabla\sigma = 0$, the motion is solenoidal or circuital, and we may write $\sigma = V\nabla\tau$ where $S\nabla\tau = 0$. τ is the vector potential of velocity. The lines $\epsilon = \frac{1}{2}V\nabla\sigma$ become in this case the concentration of $\frac{1}{2}\tau$. The lines of σ are the vortex lines of τ, and the lines of ϵ are the vortex lines of σ.

20. If σ is continuous, and the equation of a surface of discontinuity of the gradient dyadic of σ and of σ' is $f = 0$, where now σ is a displacement and σ' is $d\sigma/dt$ the velocity, we have certain conditions of kinematic compatibility. These were given by Christoffel in 1877–8 and are found as follows. We have

$$[\sigma] = 0, \qquad [- S()\nabla \cdot \sigma] = - \mu S\nu$$

in the case in which the time t is not involved; and for a moving surface in which f is a function of t as well as of ρ, we would have

$$[-S()\nabla\cdot\sigma] = -\mu S U\nabla f(),$$

$$\left[-S\frac{d\rho}{dt}\nabla\cdot\sigma\right] = -\mu S\frac{d\rho}{dt}U\nabla f = \mu\frac{df}{dt}/T\nabla f = [\sigma'] = -G\mu.$$

This gives us the discontinuity in the time rate of change of the displacement of a point as it passes from one side to the other of the moving surface. The equation of the surface as it moves being $f(\rho, t)$, we have in the normal direction

$$-Sd\rho\nabla\cdot f + dtf' = 0,$$

that is, since $d\rho$ is now $U\nabla fdn$, $dn/dt = -f'/T\nabla f = G$, where f' is the derivative of f as to t alone. In words, at any point on the instantaneous position of the moving surface the rate of outward motion of the point of the surface coinciding with the fixed point in space is $G = -f'/T\nabla f$. The moving surface of discontinuity is called a wave and G the rate of propagation of the wave at the given point. We may now read the condition of compatibility above in these words: the abrupt change in the displacement velocity is given by a definite vector μ at each point multiplied by the negative rate of propagation of the wave of displacement, that is, if G is the rate of propagation,

$$[\sigma'] = -G\mu, \qquad \text{and} \qquad [S\nabla\sigma] = -S\mu V\nabla f = -S\mu\nu.$$

21. The preceding theorem becomes general for discontinuities of any order in the following way. Let the function σ and all its derivatives be continuous down to the $(n-1)$th, then we can write

$$[S()_1\nabla\cdot S()_2\nabla\ \cdots\ S()_{n-1}\nabla\cdot\sigma] = 0,$$

whence, differentiating along the surface of discontinuity as before, we find in precisely the same manner

$$[S()_1\nabla \cdots S()_n\nabla\cdot\sigma] = \mu S()_1 U\nabla f S()_2 U\nabla f \cdots S()_n U\nabla f,$$

since at a given point *on* the fixed surface ∇f is constant. And if we insert $d\rho/dt$ in m parentheses $(m \leq n)$, we shall have, since the surface is moving,

$$[S()_1\nabla \cdots S()_{n-m}\nabla\cdot\sigma^{(m)}]$$
$$= -\mu G^m S()_1 U\nabla f \cdots S()_{n-m} U\nabla f(-1)^m.$$

In particular for $m = 2 = n$, we have

$$[\sigma''] = \mu G^2,$$

which is the discontinuity in the acceleration of the displacement.

If $m = 1$, $n = 2$,

$$[S()\nabla\cdot\sigma'] = -\mu G S() U\nabla f.$$

From this we derive easily

$$[S\nabla\sigma'] = -GS\mu U\nabla f = -GS\mu\nu.$$
$$[V\nabla\sigma'] = -GV\mu U\nabla f = -GV\mu\nu.$$

22. The nth derivatives of $S\alpha\sigma$ are

$$[S()_1\nabla \cdots S()_n\nabla\cdot S\alpha\sigma] = S()_1 U\nabla f \cdots S()_n U\nabla f S\alpha\mu.$$

If then we hold the surface fixed and consider a certain point, the discontinuity in the nth derivative of the ratio of two values of the infinitesimal volume which has two perpendicular directions on the surface and the third along the normal will be given by the formula

$$S()_1 U\nabla f \cdots S()_n U\nabla f S\mu U\nabla f.$$

In case we have a material substance that has mass and

density and of which the mass remains fixed, we have

$$c/c_0 = \mathrm{vol}_0/\mathrm{vol},$$
$$\log c - \log c_0 = \log v_0 - \log v,$$
$$\nabla \log c = -\nabla \log v/v_0 = -v_0/v\cdot\nabla(c/v_0).$$

Therefore from the formula above we have since $v_0/v = 1$ in the limit

$$[S()_1\nabla \cdots S()_n\nabla \log c] = S()_1 U\nabla f \cdots S()_n U\nabla f S\mu U\nabla f.$$

In particular for the case of discontinuities of order two we have

$$[\nabla \log c] = U\nabla f S\mu U\nabla f.$$

23. These theorems may be extended to the case in which the medium is in motion as well as the wave of discontinuity.

Stress

24. In any body the stress at a given point is given as a tension or a pressure which is exerted from some source across an infinitesimal area situated at the point. The stress real y consists of two opposing actions, being taken as positive if a tension, negative if a pressure. It is assumed that the stress taken all over the surface of an infinitesimal closed solid in the body will be a system of forces in equilibrium, to terms of the first order. This is equivalent to assuming that the stress on any infinitesimal portion of the surface is a linear function of the normal, that is

$$\Theta = \Xi U\nu.$$

25. We have therefore for any infinitesimal portion of space inside the body

$$\oint\oint \Theta dA = \oint\oint \Xi d\nu = 0.$$

But by Green's theorem this is equal to the integral through

the infinitesimal space $\int\int\int \Xi\nabla = 0$. Hence $\Xi\nabla = 0$. In this equation Ξ is a function of ρ, and ∇ differentiates Ξ.

26. In case the portion of space integrated over or through is not infinitesimal, this equation (in which Ξ is no longer a constant function) remains true if there is equilibrium; and if there are external forces that produce equilibrium, say ξ per unit volume, then the density being c, we have

$$\Xi\nabla + c\xi = 0$$

for every point.

In case there is a small motion, we have

$$\Xi\nabla + c\xi = c\sigma''.$$

27. Returning to the infinitesimal space considered, we see that the moment as to the origin of the stress on a portion of the boundary will be $V\rho\Xi U\nu$ and the total moment which must vanish, considering Ξ as constant, is

$$\oint\oint V\rho\Xi d\nu = \int\int\int V\rho\Xi\nabla dv,$$

hence

$$V\rho\Xi\nabla = 0 = \epsilon(\Xi).$$

We see therefore that Ξ is self-conjugate.

EXAMPLES

(1). Purely normal stress, hydrostatic stress. In this case Ξ is of the form $\rho\Xi = g\rho$, where g is $+$ for tension, $-$ for pressure, and is a function of ρ (scalar, of course).

(2). Simple tension or pressure.

$$\Xi = -p\alpha S\alpha.$$

(3). Shearing stress.

$$\Xi = -p(\alpha S\beta + \beta S\alpha),$$

β not parallel to α.

(4). Plane stress.

$$\Xi = g_1 \alpha S\alpha + g_2 \beta S\beta.$$

(5). Maxwell's electrostatic stress.

$$\Xi = 1/8\pi \cdot V\nabla P()\nabla P,$$

where P is the potential.

28. The quadric $S\rho\Xi\rho = -C$ is called the stress quadric. Its principal axes give the direction of the principal stresses. Since $\Xi\rho$ is the direction of the normal we may arrive at a graphical understanding of the stress by passing planes through the center, and to each construct the conjugate diameter. This will give the direction of the stress, and since $T\Xi\rho$ is inversely proportional to the perpendicular from the origin on the tangent plane at ρ, if we lay off on the conjugate diameter distances inversely as the perpendiculars, we shall have the vector representation of the stress. When the diameter is normal to its conjugate plane, there will be no component of the corresponding vector that is parallel to the plane, that is, no tangential stress. Such planes will be the principal planes of the stress.

It is evident that a stress is completely known when the self-conjugate linear vector function Ξ is known, which depends therefore upon six parameters. We shall speak, then, of the stress Ξ, since Ξ represents it. This proposition is sometimes stated as follows: stress is not a vector but a dyadic (tensor). From this point of view the six components of the stress are taken as the coordinates of a vector in six-dimensional space. These components in the quaternion notation are, for α, β, γ, a trirectangular system,

$$-S\alpha\Xi\alpha, \quad -S\beta\Xi\beta, \quad -S\gamma\Xi\gamma, \quad -S\alpha\Xi\beta = -S\beta\Xi\alpha,$$
$$-S\beta\Xi\gamma = -S\gamma\Xi\beta, \quad -S\gamma\Xi\alpha = -S\alpha\Xi\gamma.$$

That is,

$$X_x \quad Y_y \quad Z_z, \qquad X_y = Y_x, \qquad Y_z = Z_y, \qquad Z_x = X_z.$$

It is easy to see now that certain combinations of these component stresses are invariant. Thus we have at once the three invariants m_1, m_2, m_3, which are

$$X_x + Y_y + Z_z, \quad Y_yZ_z + Z_zX_x + X_xY_y - Y_z^2 - Z_x^2 - X_y^2,$$
$$X_xY_yZ_z + 2X_yY_zZ_x - X_xY_z^2 - Y_yZ_x^2 - Z_zX_y^2.$$

For any three perpendicular planes these are invariant.

EXERCISE

What are the principal stresses and principal planes of the five examples given above?

29. Returning to the equation of a small displacement, we may write it

$$\sigma'' = \xi + c^{-1}\Xi\nabla.$$

Hence the time rate of storage or dissipation of energy is

$$W' = -\int\int\int S\sigma'\Xi\nabla dv.$$

The other terms of the kinetic energy are not due to storage of energy.

Now we have an experimental law due to Hooke which in its full statement is to the effect that the stress dyadic is a linear function of the strain dyadic. The latter was shown to be

$$\varphi_0 = -\tfrac{1}{2}[S()\nabla\cdot\sigma + \nabla S\sigma()].$$

The law of Hooke then amounts to saying that Ξ is a linear function of σ and ∇ where ∇ operates upon σ, and owing to the self-conjugate character of φ, we must be able to interchange ∇ and σ, that is,

$$\Xi = \Theta[(), \nabla, \sigma].$$

First, it follows that if the strain φ_0 is multiplied by a variable parameter x, that the stress will be multiplied by the same parameter. We have then for a parametric change of this kind which we may suppose to take place in σ alone $\sigma' = \sigma x'$. Hence for a gradually increasing σ, we would have

$$W' = -xx'\int\int\int S\sigma\Xi\nabla dv,$$
$$W = -\tfrac{1}{2}\int\int\int S\sigma\Xi\nabla\, dv,$$

if x runs from 0 to 1. This gives an expression for the energy if it is stored in this special manner. If the work is a function of the strain alone and not dependent upon the way in which it is brought about, W is called an *energy-function*. It is thus seen to be a quadratic function of the strain. In case there is an energy function, we have for two strain functions due to the displacements σ_1, σ_2

$$\Xi_1 = \Theta[(), \sigma_1, \nabla_1], \qquad \Xi_2 = \Theta[(), \sigma_2, \nabla_2].$$

The stored energy for the two displacements must be the same either way we arrange the displacements, hence we have

$$S\sigma_2\Theta_3[\nabla_3, \sigma_1, \nabla_1] = S\sigma_1\Theta_4[\nabla_4, \sigma_2, \nabla_2],$$

where the subscripts 3, 4 merely indicate upon what ∇ acts. This is equivalent to saying that so far as vector function is concerned, in the form

$$S\alpha\Theta[\beta, \gamma, \delta]$$

we can interchange α, β and γ, δ. Since Ξ is self-conjugate, Θ is self-conjugate, and we can interchange α and β. From the nature of the strain function we can interchange γ, δ. Of course, in the forms above we cannot interchange the effect of the differentiations.

We have in this way arrived at six linear vector functions

$$\varphi_{11} \quad \varphi_{22} \quad \varphi_{33} \quad \varphi_{23} \quad \varphi_{31} \quad \varphi_{12},$$

wherein we can interchange the subscripts, and where

$$\varphi_{11} = \Theta[(), \alpha, \alpha] \cdots \varphi_{23} = \Theta[(), \beta, \gamma] \cdots,$$

$\alpha\, \beta\, \gamma$ being a trirectangular system of unit vectors. We have further a system of thirty-six constituents c_{1111}, c_{1112}, $\cdots$ where

$$c_{1111} = - S\alpha\varphi_{11}\alpha, \qquad c_{1112} = - S\alpha\varphi_{12}\alpha, \qquad \cdots,$$

each of the six functions having six constituents. These are the 36 elastic constants. If there is an energy function, they reduce in number to only 21, for we must be able to interchange the first pair of numbers with the last pair. There are thus left

3 forms c_{1111}, 6 of c_{1112}, 3 of c_{1122}, 3 of c_{1212}, 3 of c_{2311}, 3 of c_{2131}.

In theories of elasticity based upon a molecular theory and action at a distance six other relations are added to these reducing the number of elastic constants to 15. These relations are equivalent to an interchange of the second and third subscript in each form, thus $c_{1122} = c_{1212}$. These are usually called Cauchy's relations, but are not commonly used. (See Love, Elasticity, Chap. III.)

Remembering the strain function φ_0, we can interpret these coefficients with no difficulty, for we have

$$- S\alpha_i\varphi_0\alpha_j \cdot \varphi_{ij},$$

the stress dyadic due to the strain component $- S\alpha_i\varphi_0\alpha_j$, where α_i, α_j are any two of the three α, β, γ. c_{ijkl} is the component of the stress across a plane normal to α_j in the direction α_i due to the strain component $- S\alpha_k\varphi_0\alpha_l$.

EXAMPLES

(1). If $s_{ij} = - S\alpha_i\varphi_0\alpha_j$, show that we have for the energy function

$$W = \tfrac{1}{2}\Sigma c_{1111}s_{11}^2 + \Sigma c_{1122}s_{11}s_{22} + \tfrac{1}{2}\Sigma c_{1212}s_{12}^2 + \Sigma c_{1223}s_{12}s_{23} + \Sigma c_{1112}s_{11}s_{12} + \Sigma c_{1123}s_{11}s_{23}.$$

(2). When there is a plane of symmetry, say in the direction normal to γ, all constants that involve γ an odd number of times vanish, for the solid is unchanged by reflection in this plane. Only thirteen remain. If there are two perpendicular planes of symmetry, normal to β, γ, the only constants left are of the types

$$c_{1111}, \quad c_{1122}, \quad c_{1212},$$

the plane normal to α is thus a plane of symmetry also. There are nine constants. This last case is that of tesseral crystals.

(3). If the constants are not altered by a change of α into $-\alpha$, β into $-\beta$, as by rotation about γ through a straight angle, then the plane normal to γ is a plane of symmetry.

(4). Discuss the effect of rotation about γ through other angles.

(5). When the energy function exists we have

$$\Theta(\lambda, \mu, \nu) - \Theta(\mu, \lambda, \nu) = - V\nu\theta V\lambda\mu, \quad \text{where } \theta' = \theta.$$

30. A body is said to be isotropic as to elasticity when the elastic constants are not dependent upon directions in the body. In such case the energy function is invariant under orthogonal transformation. It must, therefore, be a function of the three invariants of φ_0, m_1, m_2, m_3. The last is of third degree, while the energy function is a quadratic and therefore can be only of the form

$$W = - Pm_1 + Am_1^2 + Bm_2.$$

P is zero except for gases and is then positive. The constant A refers to resistance to compression, and is positive. B is a constant belonging to solids.

The form given the quadratic terms by Helmholtz is

$$Am_1^2 + Bm_2 = \tfrac{1}{2}Hm_1^2 + \tfrac{1}{2}C[2m_1^2 - 6m_2].$$

The [] is the sum of the squares of the differences of the latent roots of φ_0. The constant H refers to changes of volume without change of form, and in such change it is the whole energy, for if there is no change of form, the roots are all equal and the other term is zero. C refers to changes of form without change of volume, since it vanishes if the roots are equal and is the whole energy if there is no cubical expansion m_1. For perfect fluids $C = 0$.

The form given by Kirchoff is

$$Km_1(\varphi_0^2) + K\theta m_1^2.$$

From which we have

$$B - C = 2K\theta, \quad 3C = 2K, \quad H = 2K(\theta + \tfrac{1}{3}), \quad C = \tfrac{2}{3}K.$$

We may write for solids, liquids, and gases

$$W = K\theta m_1^2 + Km_1(\varphi_0^2) - Pm_1.$$

Later notation gives $2K\theta = \lambda$, $K = \mu$, that is,

$$W = \tfrac{1}{2}\lambda m_1^2 + \mu m_1(\varphi_0^2) - Pm_1.$$

The constants λ, μ are the two independent constants of isotropic bodies.

We now have for the stress function in terms of the strain function

$$\Xi = \lambda m_1 + 2\mu\varphi_0.$$

EXAMPLES

(1). In the case of a simple dilatation we know $\Xi = p$

and we have for φ_0

$$\varphi_0 = -\tfrac{1}{2}(S()\nabla\cdot a\rho + \triangle Sa\rho()) = a().$$

Substituting in the equation above, we have

$$()\rho = \lambda(3a) + 2\mu a().$$

The cubical dilatation is thus

$$3a = p/(\lambda + \tfrac{2}{3}\mu) = p/k,$$

where k is called the modulus of cubical compression.

(2). For a simple shear.

$$\varphi_0 = -a/2\cdot[\alpha S\beta() + \beta S\alpha()], \quad m_1 = 0,$$
$$\Xi = -a\mu[\alpha S\beta() + \beta S\alpha()].$$

If the tangential stress is T, then $T = a\mu$. M is the shear modulus or simple rigidity.

(3). If a prism of any form is subject to tension T uniform over its plane ends, and no lateral traction, we have

$$\Xi = -\alpha TS\alpha() = \lambda m_1 + 2\mu\varphi_0.$$

From this equation, taking the first scalar invariant of both sides,

$$T = 3m_1\lambda + 2m_1\mu,$$

so that

$$m_1 = T/(3\lambda + 2\mu).$$

Substituting, we have

$$\varphi_0 = -\alpha\frac{T}{2\mu}S\alpha() - \frac{T\lambda}{2\mu(3\lambda + 2\mu)}.$$

We write now $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$, the quotient of a simple longitudinal tension by the stretch produced, and called Young's modulus. Also we set

$$s = \lambda/(2\lambda + 2\mu), \text{ Poisson's ratio,}$$

the ratio of the lateral contraction to the longitudinal stretch.

It is clear that if any two of the three moduli are known, the other may be found. We have

$$\lambda = E/[(1 + s)(1 - 2s), \qquad \mu = \tfrac{1}{2}E/(1 + s),$$
$$k = \tfrac{1}{3}E/(1 - 2s).$$

In terms of E and s we have

$$\varphi_0 = \left(\frac{1+s}{E}\right)\Xi\, \frac{m_1(\Xi)\cdot s}{E}.$$

(4). If $\frac{1}{2} < s, k < 0$, and the material would expand under pressure. If $s < -1$, W would not be positive.

(5). If Cauchy's relations hold, $s = \frac{1}{4}$ and $\lambda = \mu$. For numerical values of the moduli see texts such as Love, Elasticity.

31. Bodies that are not isotropic are called aelotropic. For discussion of the cases and definitions of the moduli, see texts on elasticity.

32. There is still the problem of finding σ from φ_0 after the latter has been found from Ξ. This problem we can solve as follows:

$$\begin{aligned}\sigma = \sigma_0 + \int_{\rho_0}^{\rho_1} d\sigma &= \sigma_0 - \int_{\rho_0}^{\rho_1} \sigma S\nabla d\rho, \text{ where } \nabla \text{ acts on } \sigma\\ &= \sigma_0 + \int_{\rho_0}^{\rho_1}[\varphi_0 d\rho - \tfrac{1}{2}Vd\rho V\nabla\sigma]\\ &= \sigma_0 + \int_{\rho_0}^{\rho_1}[\varphi_0 d\rho - \tfrac{1}{2}(V(\rho_1 - \rho)V\nabla d\sigma\\ &\qquad\qquad -d\cdot V(\rho_1 - \rho)V\nabla\sigma]\\ &= \sigma_0 - \tfrac{1}{2}V(\rho_1 - \rho_0)V\nabla\sigma_0 + \int_{\rho_0}^{\rho_1}[\varphi_0 d\rho\\ &\qquad\qquad - \tfrac{1}{2}V(\rho_1 - \rho)V\nabla d\sigma]\\ &= \sigma_0 - \tfrac{1}{2}V(\rho_1 - \rho_0)V\nabla\sigma_0 + \int_{\rho_0}^{\rho_1}[\varphi_0 d\rho\\ &\qquad\qquad - V(\rho_1 - \rho)V\nabla'\varphi_0' d\rho].\end{aligned}$$

We are thus able to express σ at any point ρ_1 in terms of the

values at ρ_0 of σ, $V\nabla\sigma$, and the values along the path of integration of φ_0 and $V\nabla\varphi_0()$.

EXAMPLES

(1). Let us consider a cylinder or prism which is vertical with horizontal ends, the upper being cemented to a horizontal plane. Then we have the value of

$$\Xi = -\, gc\gamma S\gamma\rho S\gamma(), \qquad \gamma \text{ vertical unit,}$$

where the origin is at the center of the lower base. The conditions of equilibrium are

$$\Xi\nabla + c\xi = 0, \qquad \text{or} \qquad c\xi = -\, gc\gamma, \qquad \xi = -\, g\gamma.$$

That is, the condition is realizable by a cylinder hanging under its own weight. The tension at the top surface is gcl where l is the length.

Solving for the strain, we have

$$\varphi_0 = \frac{gcs}{E} S\gamma\rho\cdot() + \frac{gc(1+s)}{E}\gamma S\gamma\rho S\gamma().$$

Let $a = gcs/E$, $b = gc(1+s)/E$, and note that

$$V\nabla\varphi_0() = -\, aV\gamma() - bV\gamma\gamma S\gamma() = -\, aV\gamma().$$

The integral is thus

$$\begin{aligned}
&\sigma_0 - \tfrac{1}{2}V(\rho_1 - \rho_0)\epsilon_0 \\
&\qquad + \textstyle\int_{\rho_0}^{\rho_1}[aS\gamma\rho\cdot d\rho + b\gamma S\gamma\rho S\gamma d\rho + aV(\rho_1 - \rho)V\gamma d\rho] \\
&\quad = \sigma_0 - \tfrac{1}{2}V(\rho_1 - \rho_0)\epsilon_0 \\
&\qquad\qquad + \textstyle\int_{\rho_0}^{\rho_1}[aS\gamma\rho\cdot\delta\rho + b\gamma S\gamma\rho S\gamma d\rho \\
&\qquad\qquad\qquad + a\,V\rho_1 V\gamma d\rho - ad\rho S\gamma\rho + a\gamma S\rho d\rho] \\
&\quad = \sigma_0 - \tfrac{1}{2}V(\rho_1 - \rho_0)\epsilon_0 + [\tfrac{1}{2}b\gamma S^2\gamma\rho \\
&\qquad\qquad\qquad + aV\rho_1 V\gamma\rho + \tfrac{1}{2}a\gamma\rho^2]_{\rho_0}^{\rho_1},
\end{aligned}$$

the differential being exact. This gives us as the value of σ at ρ_1,

$$\sigma_1 = \zeta + V(\rho_1 - \rho_0)(\tfrac{1}{2}\epsilon_0 + aV\gamma\rho_0) + \tfrac{1}{2}aV\rho_1\gamma\rho_1 + \tfrac{1}{2}b\gamma S^2\gamma\rho, \qquad \zeta,\ \epsilon \text{ constants.}$$

Substituting a and b, and constructing

$$\varphi_0 = -\tfrac{1}{2}[S()\nabla\cdot\sigma + \nabla S\sigma()],$$

we easily verify. If the cylinder does not rotate, we may omit the second term and if the upper base does not move laterally, then the vector ζ reduces to $-\tfrac{1}{2}gcl^2/E\cdot\gamma$, and we have

$$\sigma = -\tfrac{1}{2}gcl^2/E\cdot\gamma + gcs/2E\cdot V\rho\gamma\rho + gc(1+s)/2E\cdot\gamma S^2\rho\gamma.$$

A plane cross-section of the cylinder is distorted into a paraboloid of revolution about the axis and the sections shrink laterally by distances proportional to their distances from the free end.

(2). If a cylinder of length $2l$ is immersed in a fluid of density c', its own density being c, the upper end fixed, p the pressure of the fluid at the center of gravity, then we have the stress given by

$$\Xi = -(\mathrm{p} + gc'S\gamma\rho) - g(c - c')(1 - S\gamma\rho)\gamma S\gamma,$$

whence calculating φ_0, we have

$$\varphi_0 = 1/E\cdot[-(p + gc'S\gamma\rho)(-1 + 2s) - gs(c - c') \times (1 - S\gamma\rho)] - \gamma S\gamma[g(c - c')(1 - S\gamma\rho)l + s)]/E.$$

And

$$\sigma = \zeta + V\theta\rho + \rho[(-1 + 2s)p - gsl(c - c') - S\rho\gamma g[c\epsilon - s(c + c')]/E + \gamma[\tfrac{1}{2}g(c - c')(1 + s)(1 - S\gamma\rho)^2 + \tfrac{1}{2}g\rho^2[c' - s(c + c')]/E.$$

(3). What does the preceding reduce to if $c = c'$? Solve also directly.

(4). If a circular bar has its axis parallel to γ, and the only stress is a traction at each end, equivalent to couples of moment $\frac{1}{2}\pi a^4 \mu t$, about the axis of γ, a being the radius, that is, a round bar held twisted by opposing couples, we have

$$\begin{aligned} \Xi &= -\tfrac{1}{2}\mu t(\gamma S\rho\gamma() + V\rho\gamma S\gamma()), \\ \varphi_0 &= -\tfrac{1}{4}t[\gamma S\rho\gamma() + V\rho\gamma S\gamma()], \\ \sigma &= tV\rho\gamma S\gamma\rho. \end{aligned}$$

Any section is turned in its own plane through the angle $-tS\gamma\rho$. t is the angular twist per centimeter.

(5). The next example is of considerable importance, as it is that of a bar bent by couples. The equations are

$$\begin{aligned} \Xi &= -E/R\cdot S\alpha\rho\cdot\gamma S\gamma(), \\ \varphi_0 &= -(1+s)/R\cdot S\alpha\rho\cdot\gamma S\gamma() - s/R\cdot S\alpha\rho\cdot(), \\ \sigma &= \tfrac{1}{2}R^{-1}\cdot\alpha[S^2\gamma\rho + sS^2\alpha\rho - sS^2\gamma\alpha\rho] \\ &\qquad + sR^{-1}\gamma\alpha S\beta\rho S\alpha\rho - R^{-1}\gamma S\alpha\rho S\gamma\rho. \end{aligned}$$

If the body is a cylinder or prism of any shape with the axis γ horizontal, there is no body force nor traction on the perimeter. The resultant traction across any section is

$$\iint -E/R\cdot S\alpha\rho\, dA,$$

which will equal zero if the origin is on the line of centroids of the sections in the normal state, that is, the neutral axis. Thus the bar is stressed only by the tractions at its terminal sections, the traction across any section being equivalent to a couple.

The couple becomes one with axis $\beta = \gamma\alpha$ and value EI/R, where I is the moment of inertia about an axis through the centroid parallel to β. The line of centroids is displaced according to the law

$$-S\alpha\sigma = \tfrac{1}{2}R^{-1}S^2\gamma\rho,$$

so that it is approximately the arc of a circle of radius R. The strain-energy function is $\frac{1}{2}ER^{-2}\cdot S^2\alpha\rho$, and the potential energy per unit length $\frac{1}{2}EI/R^2$.

For further discussion see Love, p. 127 et seq.

(6). When $\Xi = -E\cdot S\gamma\rho\cdot\theta()$, where $\theta\gamma = 0$, and $\theta = \theta'$, and α may not be a unit vector, show that

$$\begin{aligned}\varphi_0 &= -(1+s)S\gamma\rho\cdot\theta() + sS\gamma\rho\cdot m_1(\theta),\\ \sigma &= (1+s)[\tfrac{1}{2}S\rho\theta\rho - \theta\rho S\rho\gamma] + sm_1(\theta)[-\tfrac{1}{2}\gamma\rho^2 + \rho S\rho\gamma].\end{aligned}$$

See Love, pp. 129–130.

33. We recur now to the equation of equilibrium

$$\Xi\nabla + c\xi = 0.$$

In this we substitute the value of

$$\Xi = \lambda m_1 + 2\mu\varphi_0 = -\lambda S\nabla\sigma - (\sigma S()\nabla + \nabla S\sigma()),$$

whence

$$\lambda\nabla S\nabla\sigma + \mu\nabla^2\sigma + \mu\nabla S\nabla\sigma - c\xi = 0,$$

or

$$(\lambda+\mu)\nabla S\nabla\sigma + \mu\nabla^2 c - c\xi = 0,$$

or equally since

$$\begin{aligned}\nabla^2\sigma &= \nabla S\nabla\sigma + \nabla V\nabla\sigma,\\ (\lambda+2\mu)\nabla S\nabla\sigma &+ \mu\nabla V\nabla\sigma - c\xi = 0.\end{aligned}$$

This is the equation of equilibrium when the displacement and the force ξ are given. In the case of small motion we insert on the right side instead of 0, $-c\sigma''$. The traction across a plane of normal ν is

$$-(\lambda+\mu)\nu S\nabla\sigma - \mu V\nabla\nu\sigma,$$

where ν is constant. Operating on the equilibrium equation by $S\nabla()$, we see that

$$(\lambda+2\mu)\nabla^2 S\nabla\sigma - cS\nabla\xi = 0.$$

If then there are no body forces ξ or if the forces ξ are derivable from a force-function P and $\nabla^2 P = 0$ throughout the body, we see that

$$S\nabla\sigma$$

is a harmonic function. Since $m_1(\Xi) = 3km_1$, we see that $m_1(\Xi)$ is also harmonic.

Again we have

$$(\lambda + \mu)\nabla S\nabla\sigma = -\mu\nabla^2\sigma,$$

whence we can construct the operators

$$(\lambda + \mu)\nabla S\nabla()S\nabla\sigma = -\mu\nabla^2\nabla S\sigma = -\mu\nabla^2\sigma S\nabla().$$

and adding the two,

$$2(\lambda + \mu)\nabla S\nabla S\nabla\sigma() = -\mu\nabla^2(\sigma S\nabla() + \nabla S\sigma())$$

Now we have

$$\Xi = -\lambda S\nabla\sigma - \mu(\sigma S\nabla() + \nabla S\sigma()),$$

and since $S\nabla\sigma$ is harmonic

$$\nabla^2\Xi = -\mu\nabla^2(\sigma S\nabla() + \nabla S\sigma()) = 2(\lambda + \mu)\nabla S\nabla S\nabla\sigma()$$

$$= \frac{2(\lambda + \mu)}{3k}\nabla S\nabla S\nabla\sigma() = (1 + s)^{-1}\nabla S\nabla S\nabla\sigma().$$

or

$$\nabla^2\Xi = \frac{1}{1+s}\nabla S\nabla M_1().$$

This relation is due to Beltrami, R. A. L. R., (5) 1 (1892).

EXAMPLE

Maxwell's stress system cannot occur in a solid body which is isotropic, free from the action of body forces, and slightly strained from a state of no stress, since we have

$$-m_1(\Xi) = 1/8\pi\cdot(\nabla P)^2,$$

which is not harmonic. (Minchin Statics, 3d ed. (1886), vol. 12, ch. 18.)

34. We consider now the problem of vibrations of a solid under no body forces, the body being either isotropic or aeolotropic.

The equation of vibrations is

$$c\sigma'' = \Theta(\nabla, \nabla, \sigma), \quad \text{where} \quad \Xi = \Theta[(), \nabla, \sigma] \text{ as before, and}$$

σ is a function of both t and ρ. If the vector ω represents the direction and the magnitude of the wave-front, the equation of a plane-wave will be

$$u = t - S\rho/\omega,$$

since this represents a variable plane moving along its own normal with velocity ω. By definition of a wave-front the displacement from the mean position is at any instant the same at every point. That is, σ is a function of u and t, hence

$$\nabla\sigma = -\nabla S\rho/\omega\partial\sigma/\partial u = \omega^{-1}\partial\sigma/\partial u,$$

and any homogeneous function of ∇ as $f(\nabla)$ gives

$$f\nabla\cdot\sigma = f(\omega^{-1})\partial^n\sigma/\partial u^n,$$

where n is the degree of f.

The equation above for wave-motion then is

$$c\sigma'' = \Theta[\omega^{-1}, \omega^{-1}, \partial^2\sigma/\partial u^2].$$

If the wave is permanent, σ involves t only through u and if the vibration is harmonic of frequency p,

$$\sigma'' = \partial u^2\sigma/\partial^2 = -p^2\sigma.$$

Therefore

$$\Theta[U\omega, U\omega, \sigma] = c\sigma T^2\omega.$$

Hence for a plane wave propagated in the direction $U\omega$

the vibration is parallel to one of the invariant lines of the function

$$\Theta[U\omega, U\omega, ()].$$

The velocity is the square root of the quotient of the latent root corresponding, by the density. There may be three plane-polarized waves propagated in the same direction with different velocities. The wave-velocity surface is determined by the equation

$$S[\Theta(\omega^{-1}, \omega^{-1}, \alpha) - c\alpha][\Theta(\omega^{-1}, \omega^{-1}, \beta - c\beta][\Theta(\omega^{-1}, \omega^{-1}, \gamma] = 0,$$

that is, by the cubic of $\Theta[U\omega, U\omega, ()]$.

If there is an energy function, $\Theta[U\omega, U\omega, ()]$ is self-conjugate as may easily be seen. In such case the invariant lines are perpendicular, that is, the three directions of vibration, θ_1, θ_2, θ_3, for any direction of propagation are mutually trirectangular. Since W is essentially positive, the roots are positive, and there are thus three real velocities in any direction.

If g is a repeated root, there is an invariant plane of indeterminate lines and the condition for such is

$$V[\Theta(\omega^{-1}, \omega^{-1}, \alpha) - c\alpha][\Theta(\omega^{-1}, \omega^{-1}, \beta) - c\beta] = 0,$$

α and β arbitrary. There is a finite number of solutions to this vector equation, giving ω, and these give Hamilton's internal conical refraction. The vectors terminate at double points of the wave-velocity surface.

The index-surface of MacCullagh, that is, Hamilton's wave-slowness surface, is given by

$$S[\Theta(\rho, \rho, \alpha) - c\alpha][\Theta(\rho, \rho, \beta) - c\beta][\Theta(\rho, \rho, \gamma) - c\gamma] = 0,$$

α, β, γ arbitrary, which is the inverse of the wave-velocity surface. ρ is the current vector of the surface, just as ω for the other surface, the equation being formed by setting

$\rho = -\omega^{-1}$. The wave-surface, or surface of ray-velocity, is the envelope of $S\rho/\omega = 1$, or $S\rho\mu = -1$, where $\mu = -\omega^{-1}$. The condition is that given by the equations of the two other surfaces. It is the reciprocal of the index surface with respect to the unit sphere $\rho^2 = -1$, or the envelope of the plane wave-fronts in unit time after passing the origin, or the wave of the vibration propagated from the origin in unit time. The vectors ρ that satisfy its equation are in magnitude and direction the ray-velocities. When there is an energy function, this ray-velocity is found easily, as follows:

The wave-surface is the result of eliminating between

$$\Theta(\mu, \mu, \sigma) = c\sigma,$$

$$\Theta(d\mu, \mu, \sigma) + \Theta(\mu, d\mu, \sigma) + \Theta(\mu, \mu, d\sigma) = cd\sigma,$$
$$S\mu\rho = -1 \cdot S\rho d\mu = 0.$$

From the second equation

$$2Sd\mu\Theta(\sigma, \sigma, \mu) + Sd\sigma\Theta(\mu, \mu, \sigma) = cS\sigma d\Sigma,$$

or by the equations

$$Sd\mu\Theta(\sigma, \sigma, \mu) = 0.$$

Hence as $d\mu$ is perpendicular to ρ, we have

$$\Theta(\sigma, \sigma, \mu) = x\rho.$$

Operate by $S\mu$ and substitute the value of x,

$$\Theta(U\sigma, U\sigma, \mu) = c\rho.$$

This equation with $\Theta(\mu, \mu, \sigma) = c\sigma$ gives all the relations between the three vectors. See Joly, p. 247 et seq.

CHAPTER XI

HYDRODYNAMICS

1. Liquids and gases may be considered under the common name of fluids. By definition, a perfect fluid as distinguished from a viscous fluid has the property that its state of stress in motion or when stationary can be considered to be an operator which has three equal roots and all lines invariant, thus

$$\Xi = -p\,(\,),$$

where p is positive, that is, a pressure, or $\Xi = -p$. If the density is c, we have, when there are external forces and motion, the fundamental equation of hydrodynamics

$$\sigma'' = \xi - c^{-1}\nabla p.$$

In the case of viscous fluids we have to return to the general equation

$$c\,(\sigma'' - \xi) = -\nabla p - (\lambda + \mu)\,\nabla S \nabla\,\sigma - \mu\nabla^2\sigma.$$

2. When there is equilibrium

$$\nabla p = c\xi.$$

If the external forces may be derived from a force function, P, we have $\nabla p = c\nabla P$, hence $-Sd\rho\nabla p = -Scd\rho\nabla P$, or $dp = cdP$ for all directions. That is, any infinitesimal variation of the pressure is equal to the density into the infinitesimal variation of the force function. In order that there may be equilibrium under the forces that reduce to ξ, we must have ξ subject to a condition, for from $\nabla p = c\xi$, we have $\nabla^2 p = \nabla c\xi + c\nabla\xi$, whence $S\xi\nabla\xi = 0$, and $V\nabla\xi = V\xi\nabla \log c$.

If $\xi = \nabla P$, the condition is, of course, satisfied, and from the last equation we see that ξ is parallel to ∇c, that is to say, ξ is normal to the isopycnic surface at the point, or the levels of the force function are the isopycnic surfaces. The equation $\nabla p = c\xi$ states that ξ is also a normal of the isobaric surfaces. In other words, in equilibrium the isobaric surfaces, the isopycnic surfaces, and the isosteric surfaces are geometrically the same. However, it is to be noted that if a set of levels be drawn for any one of the three so that the values of the function represented differ for the levels by a unit, that is, if unit sheets are constructed, then the levels in the one case may not agree with the levels in the other two cases in distribution.

The fundamental equation above may be read in words: *the pressure gradient is the force per unit volume. Specific volume times pressure gradient is the force per unit mass.*

We can also translate the differential statement into words thus: *the mean specific volume in an isobaric unit sheet is the number of equipotential unit sheets that are included in the isobaric unit sheet. The average density in an equipotential unit sheet is the number of isobaric unit sheets enclosed.*

Since dp and dP are exact differentials, we have:

Under statical conditions the line integral of the force of pressure per unit mass as well as the line integral of the force from the force function per unit volume are independent of the path of integration and thus depend only on the end points.

3. There is for every fluid a characteristic equation which states a relation between the pressure, the density, and a third variable which in the case of a gas may be the temperature, or in the case of a liquid like the sea, the salinity. Thus the law of Gay-Lussac-Mariotte for a gas is

$$p = \text{const} \cdot c\,(1 + \tfrac{1}{273}\,T) \text{ for constant volume.}$$

The characteristic equation usually appears in the form $pa = RT$, where in this case a is the specific volume, the equation reading

$$dP = a dp.$$

From this we have

$$dP = RT dp/p.$$

If T is connected with p by any law such as that given above, we can substitute its value and integrate at once. Or if T is connected with the force function P by an equation, we can integrate at once.

Example.

In the case of gravity and the atmosphere, suppose that the temperature decreases uniformly with the equipotentials. Since we must in this case take P so that ∇P will be negative, we have

$$dP = -RT dp/p, \qquad T = T_0 - bP,$$

whence

$$dP = -dT/b, \qquad dT/T = Rb\, dp/p, \qquad T = T_0(p/p_0)^{bR}.$$

Or again

$$dP/(T_0 - bP) = -R\, dp/p, \qquad 1 - bP/T_0 = (p/p_0)^R.$$

We thus have the full solution of the problem, the initial conditions being for mean sea-level, and in terms of a or c as follows:

$$T = T_0(p/p_0)^{bR}, \qquad a = a_0\,(p/p_0)^{bR-1},$$

$$P = b^{-1}T_0[1 - (p/p_0)^R],$$

$$T = T_0(1 - bT_0{}^{-1}P), \qquad c = c_0(1 - T_0{}^{-1}bP)^{R^{-1}b^{-1}-1},$$

$$p = p_0(1 - T_0{}^{-1}bP)^{b^{-1}R^{-1}}.$$

Absolute zero would then be reached at a height where the

gravity potential would be

$$P = T_0/b,$$

and substituting we find $c = 0$, $p = 0$. If b is negative, the fictive limit of the atmosphere is below sea-level. For values of bR from ∞ to 1, for the latter value $b = 0.00348$ (that is, a temperature drop of 3.48° C. per 100 dynamic meters of height), we have unstable equilibrium, since from the equations above for c we have increasing density upwards. The case $bR = 1$ is extreme; however, it is mathematically interesting from the simplicity that results. Pressure and temperature would decrease uniformly and we should have a homogeneous atmosphere. This condition is unstable and the slightest displacement would continue indefinitely. Values of bR less than 1 lead still to unstable equilibrium, the state of indifferent equilibrium occurring when the adiabatic cooling of an upward moving mass of air brings its temperature to that of the new levels. For dry air this occurs for $bR = 0.2884 = (1.4053 - 1)/1.4053$, or a fall of 1.0048° C. per dynamic hectometer.

See Bjerknes, *Dynamic Meteorology and Hydrography.*

4. The equation when there is not equilibrium gives us

$$a\nabla p - \xi = -\sigma''.$$

Let $\xi = \nabla P$, and operate by $V\cdot\nabla\ (\ )$, then

$$V\nabla a\nabla p = -V\nabla\sigma''.$$

If we multiply by $SU\nu$ and integrate over any surface normal to $U\nu$, we have

$$\int\int SU\nu V\nabla a\nabla p = -\int\int SU\nu\nabla\sigma'' = -\oint Sd\rho\sigma''.$$

The right-hand side is the circulation of the acceleration or force per unit mass around any loop, the left-hand side

is the surface integral of $V\nabla a\nabla p$ over the area enclosed. If then we suppose that in a drawing we represent the isobars as lines, and the isosterics also as lines that cut these, drawing a line for the level that bounds a unit sheet in each case (and noticing that in equilibrium the lines do not intersect), we shall have a set of curvilinear parallelograms representing tubes. The circulation of the force per unit mass around any boundary will then be the number of parallelograms enclosed. It is to be noticed that the areas must be counted positively and negatively, that is, the number of tubes must be taken positive or negative, according to whether ∇a, ∇p, the two gradients, make a positive or a negative angle with each other in the order as written. This circulation of the force per unit mass may be taken as a measure of the departure from equilibrium. In the same way we find that if we draw the equipotentials and the isopycnics, we shall have the number (algebraically considered) of unit tubes in any area equal to the circulation of the force per unit volume around the bounding curve.

If we choose as boundary, for example, a vertical line, an isobaric curve, a downward vertical, and an isobaric curve, the number of isobaric-isosteric tubes enclosed gives the difference between the excess up one vertical of the cubic meters per ton at the upper isobar over that at the lower isobar and the corresponding excess for the other vertical. If the lines are two verticals and two equipotentials, the number of isopotential-isopycnic tubes is the difference of the two excesses of pressure at the lower levels over pressure at the upper levels. These are the circulations around the boundaries of the forces per unit mass or unit volume as the case may be.

5. If we integrate the pressure over a closed space inside

the fluid, we have

$$\oint\!\!\oint pU\nu dA = \int\!\!\int\!\!\int \nabla p dv = \int\!\!\int\!\!\int c\xi dv.$$

But this latter integral is the total force on the volume enclosed. This is Archimedes' principle, usually related to a body immersed in water, in which case the statement is that the resultant of all the pressure of the water upon the immersed body is equal to the weight of the water displaced. If we were to consider the resultant moment of the normal pressures and the external forces, we would arrive at an analogous statement. The field of force, however, need not be that due to gravity.

EXERCISE.

Consider the case of a field in which there is the vertical force due to gravity and a horizontal force due to centrifugal force of rotation.

6. We turn our attention now to moving fluids. A small space containing fluid with one of its points at ρ_0 may be followed as it moves with the fluid, always containing the same particles. It will usually be deformed in shape. The position ρ of the particle initially at ρ_0 will be a function of ρ_0 and of t, say

$$\rho = \theta\,(\rho_0, t).$$

The particle initially at $\rho_0 + d\rho$ will at the same time t arrive at the position

$$\rho + d_1\rho = \theta\,(\rho_0 + d\rho, t) = \rho - Sd_1\rho\nabla_0 \cdot \rho,$$

hence $d_1\rho$ becomes at time t

$$- Sd_1\rho\nabla_0 \cdot \rho = \varphi d_1\rho_0.$$

It follows that the area $Vd_1\rho d_2\rho = V\varphi d_1\rho_0\varphi d_2\rho_0$, and the

volume

$$- Sd_1\rho d_2\rho d_3\rho = - S\varphi d_1\rho_0\varphi d_2\rho_0\varphi d_3\rho_0 = - Sd_1\rho_0 d_2\rho_0 d_3\rho_0 \cdot m_3(\varphi).$$

If the fluid has a constant mass, then we must have

$$cdv = c_0 dv_0, \quad \text{or} \quad cm_3 = c_0.$$

This is the *equation of continuity in the Lagrangian form.* The reference of the motion to the time and the initial configuration is usually called reference to the Lagrangian variables.

7. Since

$$dp = - Sd\rho\nabla p = - S\varphi d\rho_0 \nabla p = - Sd\rho_0 \varphi' \nabla p = - Sd\rho_0 \nabla_0 p,$$

$$\nabla_0 p = \varphi' \nabla p = - \nabla_0 S\rho\nabla \cdot p.$$

But the equations of motion are already given in the form

$$a\nabla p = \xi - \rho'',$$

hence in terms of the variables ρ_0 and t we have

$$a\nabla_0 p = \varphi'(\rho - \rho'').$$

This equation, the characteristic equation of the fluid

$$F(p, c, T) = 0,$$

and the equation of continuity, give us five scalar equations expressing six numbers in terms of ρ_0 and t. In order to make any problem definite then, we must introduce a further hypothesis. The two that are the most common are

(1) The temperature is constant, if T is temperature, or the salinity is constant, if T is salinity. In case both variables come in, we must have two corresponding hypotheses;

(2) The fluid is a gas subject to adiabatic change. The relation of pressure to density in this case is usually written

$$p = kc^{\gamma}.$$

γ is the ratio of specific heat under constant pressure to that under constant volume, as for example, for compressed air, $\gamma = 1.408$.

8. In the integrations we are obliged to pay attention to two kinds of conditions, those due to the initial values of the space occupied by the fluid at $t = 0$, the pressure p_0 and density c_0, or specific volume a_0, at each point of the fluid, and the initial velocities of the particles ρ_0' at ρ_0. The other conditions are the boundary conditions during the movement. As for example, consider a fluid enclosed in a tank or in a pipe or conduit. The velocity in the latter case must be tangent to the walls. If we have the general case of a moving boundary for the fluid, then its equation would be

$$f(\rho, t) = 0.$$

If then ρ' is the velocity, we must have

$$-S d\rho \nabla f + (\partial f/\partial t)dt = 0, \quad \text{or} \; - S\rho' \nabla f + \partial f/\partial t = 0.$$

If there is a free surface, then the pressure here must be constant, as the pressure of the air. In order to have various combinations of these conditions coexistent, it is necessary sometimes to introduce discontinuities.

9. If we were in a balloon in perfect equilibrium moving along with one and the same mass of air, the barograph would register the varying pressures on this mass, the thermograph the varying temperatures, and if there were a velocitymeter, it would register the varying velocity of the mass. From these records one could determine graphically or numerically the rates of change of all these quantities as

they inhere in the same mass. That is, we would have the values of

$$dp/dt, \qquad dT/dt, \qquad d\rho/dt.$$

These may be called the individual time-derivatives of the quantities. As the balloon passed any fixed station the readings of all the instruments would be the same as instruments at the fixed stations. But the rates of change would differ. The rates of change of these quantities at the same station would be for a fixed ρ and a variable t, and could be called the local time-derivatives, or partial derivatives. They can be calculated from the registered readings. The relation between the two is given by the equation

$$d/dt = \partial/\partial t - S\rho'\nabla.$$

Thus we have between the individual and the local values the relations

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} - S\rho'\nabla p, \quad \frac{dT}{dt} = \frac{\partial T}{\partial t} - S\rho'\nabla \cdot T,$$

$$\frac{d\rho'}{dt} = \frac{\partial \rho'}{dt} - S\rho'\nabla \cdot \rho'.$$

The last equation gives us the individual acceleration in terms of the local acceleration and the velocity. From the fundamental equation we have

$$a\nabla p = \xi - \partial\rho'/\partial t + S\rho'\nabla \cdot \rho' = \xi - \partial\rho'/\partial t - \theta(\rho'),$$

where the function

$$\theta = -S(\,)\nabla \cdot \rho', \qquad \theta' = -\nabla S\rho'(\,), \qquad \theta_0 = \tfrac{1}{2}(-S(\,)\nabla \cdot p' - \nabla S\rho'(\,)),$$

$$2\epsilon = V\nabla\rho'.$$

This statement of the motion in terms of the coordinates of

any point and the time is the statement in terms of *Euler's variables.*

Since near ρ_0, $\rho = \rho_0 + \rho_0' dt$, we have the former function φ at this point in the form

$$\varphi = -S(\,)\nabla_0 \cdot \rho = 1 + dt(-S(\,)\nabla \cdot \rho')_0 = 1 + dt\theta \text{ at } \rho_0.$$

Whence

$$m_3(\varphi) = 1 + dtm_1(\theta) = 1 + dt(-S\nabla\rho').$$

Since the initial point is any point, this equation holds for any point and we have the equation of continuity in the form

$$c - cdtS\nabla\rho' = c_0 = c_0 + dt \cdot dc/dt(1 - dtS\nabla\rho'),$$

or, dropping terms of second order,

$$dc/dt - cS\nabla\rho' = 0.$$

This is the equation of continuity in the Euler form. If we use local values,

$$\partial c/\partial t - S\nabla(c\rho') = 0.$$

That is, the local rate of change of the density is the convergence of specific momentum. It is obvious that if the fluid is incompressible, that is, if the density is constant, then the velocity is solenoidal. If the specific volume at a local station is constant, then the specific momentum is solenoidal. If the medium is incompressible and homogeneous, then both velocity and specific momentum are solenoidal vectors. It is clear also that in any case the normal component of velocity must be continuous through any surface, but specific momentum need not be. If any boundary is stationary, then both velocity and specific momentum are tangential to it.

In the atmosphere, which is compressible, specific momentum is solenoidal, but in the incompressible hydrosphere, both velocity and specific momentum are solenoidal. Of course the specific volume of the air changes at a station, but only slowly, so that the approximate statement made is close enough for meteorological purposes.

If at any given instant we draw at every point a vector in the direction of the velocity, these vectors will determine the vector lines of the velocity which are called lines of flow. These lines are not made up of the same particles and if we were to mark a given set of particles at any time, say by coloring them blue, then the configuration of the blue particles would change from instant to instant as they moved along. The trajectory of a blue particle is a stream line. If the particles that pass a given point are all colored red, then we would have a red line as a line of flow, only when the condition of the motion is that called stationary. In this case the line through the red particles would be the streamline through the point. If the motion is not stationary, then after a time the red particles would form a red filament that would be tangled up with several stream lines.

10. In the case of meteorological observations the direction of the wind is taken at several stations simultaneously and by the anemometer its intensity is given. These data give us the means of drawing on a chart suitably prepared the lines of flow at the given time of day and the curves showing the points of equal-intensity of the wind velocity. Of course, the velocity is usually only the horizontal velocity and the vertical velocity must be inferred.

One of the items needed in meteorological and other studies is the amount of material transported. If the specific momentum in a horizontal direction is $c\rho'$, and lines

of flow be drawn, then for a vertical height dz and a width between lines of flow equal to dn, we will have the transport equal to $T\rho' dndz$. Since, however, we have for practical purposes $dz = -dp$, we can write this in the form

$$\text{transport} = T\rho' dn(-dp).$$

In order to do this graphically we first draw the lines of flow and the intensity curves. An arbitrary outer boundary curve is then divided into intervals of arc such that the projection of an interval perpendicular to the nearest lines of flow multiplied by the value of $T\rho'$ is a constant. Through these points a new set of lines of flow is constructed. The transport between these lines is then known horizontally for a constant pressure drop, by drawing the intensity curves that represent $T\rho' dn$, and if these are at unit values of the transport, they will divide the lines of flow into quadrilaterals such that the amount of air transported horizontally decreases or increases by units, and thus the vertical transport must respectively increase or decrease by units, through a sheet whose upper and lower surfaces have pressure difference equal to $dp = -1$. Towards a center of convergence the lines of flow approach indefinitely close. dn decreases and it is clear that the vertical transport upward increases. There may be small areas of descending motion, however, even near such centers. In this manner we may arrive at a conception of the actual movement of the air.

Since the specific momentum is solenoidal, we can ascertain its rate of change vertically from horizontal data. For

$$0 = S\nabla c\rho' = -\partial Z/\partial z + \text{horizontal convergence},$$

or

$$\partial Z/\partial z = \text{horizontal convergence of specific momentum}.$$

Substituting the value of dz, we have

$$\partial Z / (-\partial p) = \text{horizontal convergence of velocity},$$
$$\partial Z/\partial p = \partial T\rho'/\partial s + T\rho'\delta.$$

where ∂s runs along the lines of flow, and δ is the divergence per unit ∂s of two lines of width apart equal to 1.

These considerations enable us to arrive at the complete kinematic diagnosis of the condition of the air. On this is based the prognostications.

11. When the density c is a function of the pressure p, and the forces and the velocities can be expressed as gradients, then we have a very simple general case. Thus let

$$c = f(p), \qquad \xi = \nabla u(\rho, t), \qquad \rho' = \nabla v(\rho, t),$$

and set

$$Q = u - \int a dp, \text{ then } \nabla Q = \xi - a\nabla \rho,$$

the equations of motion are

$$\partial\rho'/\partial t + \theta(\rho') = \nabla Q, \text{ or since } \rho' = \nabla v,$$
$$\nabla[\partial v/\partial t + \tfrac{1}{2}T^2\nabla v - Q] = 0.$$

Hence the expression in brackets is independent of ρ and depends only on t and we have

$$\partial v/\partial t + \tfrac{1}{2}T^2\nabla v - Q = h(t).$$

We could, however, have used for v any function differing from v only by a function of t, thus we may absorb the function of the right into v and set the right side equal to zero. We thus have the equations of motion

$$\partial v/\partial t + \tfrac{1}{2}T^2\nabla v - Q = 0, \qquad \partial c/\partial t - S\nabla(c\nabla v) = 0,$$
$$c = f(p).$$

From these we have v, c, p in terms of ρ and t.

12. In the case of a permanent motion, the tubes of flow are permanent. If we can set $\xi = \nabla u(\rho)$, then we place

$Q = u - \int a dp$, and noticing that ρ' and Q do not depend on t, we have

$$S\rho'\nabla \cdot p' = -\nabla Q.$$

If we operate by $-Sd\rho = -S(dsU\rho')$, we have $dsSU\rho'T\rho'\nabla T\rho'$ on the left, since $S\rho'\nabla \cdot U\rho' = 0$. Hence from this equation we have at once

$$-Sd\rho(\tfrac{1}{2}T^2\rho' - Q) = 0.$$

Hence along a tube of flow of infinitesimal cross-section

$$\tfrac{1}{2}T^2\rho' - Q = C.$$

This is called *Bernoulli's theorem.* C is a function of the two parameters that determine the infinitesimal line of flow. Hence along the same tube of flow

$$\tfrac{1}{2}(T^2\rho' - T^2\rho_0') = Q - Q_0 = u - u_0 - \int_{p_0}^{p} a dp.$$

In the case of a liquid a is constant and we can integrate at once, giving

$$\tfrac{1}{2}T^2\rho' - u + ap = C.$$

From this we can find the velocity when the pressure is given or the pressure when the velocity is given. Since the pressure must be positive, it is evident that the velocity square $\leqq 2(u + C)$, or else the liquid will separate. This fact is made use of in certain air pumps. In the case of no force but gravity we have $u = gz$,

$$\tfrac{1}{2}T^2\rho' - gz + ap = C.$$

This is the fundamental equation of hydraulics. We cannot enter upon the further consideration of it here.

Vortices.

13. In the case of $\rho' = \nabla v$ it is evident that $V\nabla\rho' = 0$. When this vector, or the vector ϵ (§ 9) does not vanish,

there is not a velocity potential and vortices are said to exist in the fluid. It is obvious that if a particle of the fluid be considered to change its shape as it moves, then ϵ is the instantaneous velocity of rotation. At any instant all the vortices will form a vector field whose lines have the differential equation

$$Vd\rho V\nabla\rho' = 0 = Sd\rho\nabla' \cdot \rho - \nabla S\rho' d\rho;$$

that is,

$$\theta' d\rho = d\rho', \qquad \text{or} \qquad \theta'\rho' = d\rho'/dt,$$

from which

$$\rho' = e^{\int \theta' dt}\rho_0'.$$

These vector lines are called the vortex lines of the fluid. Occasionally the vortex lines may be closed, but as a rule the solutions of such a differential equation as the above do not form closed lines, in which case they may terminate on the walls of the containing vessel, or they may wind about indefinitely. The integral of this equation will usually contain t, and the vortices then vary with the time, but in a stationary motion they will depend only upon the point under consideration.

14. The equations of motion may be expressed in terms of the vortex as follows, since

$$V\rho' V\nabla\rho' = S\rho'\nabla\cdot\rho' - \tfrac{1}{2}\nabla\rho'^2,$$

we have

$$S\rho'\nabla\cdot\rho' = 2V\rho'\epsilon + \tfrac{1}{2}\nabla\rho'^2,$$

and thus

$$a\nabla\rho = \xi - \partial\rho'/\partial t + \tfrac{1}{2}\nabla\rho'^2 + 2V\rho'\epsilon.$$

15. When now $\xi = \nabla u(\rho, t)$, and $c = f(p)$, we set $P = \int a dp$, giving $\nabla P = a\nabla p$, and thence

$$\nabla P = \nabla u - \partial\rho'/\partial t + \tfrac{1}{2}\nabla\rho'^2 - 2V\epsilon\rho'.$$

Or, if we set $H = u + \frac{1}{2}\rho'^2 - P$, we have

$$\partial\rho'/\partial t + 2V\epsilon\rho' = \nabla H.$$

Operate on this with $V\cdot\nabla(\,)$, and since $V\nabla\partial\rho'/\partial t = 2\partial\epsilon/\partial t$, and $V\nabla V\epsilon\rho' = S\epsilon\nabla\cdot\rho' - \epsilon S\nabla\rho' - S\rho'\nabla\cdot\epsilon$, $\partial\epsilon/\partial t - S\rho'\nabla\cdot\epsilon = d\epsilon/dt$, $S\nabla\rho'$ by the continuity equation is equal to $c^{-1}\partial c/\partial t = -\,a^{-1}\partial a/\partial t$, we have

$$d(a\epsilon)/dt = -\,S(a\epsilon)\nabla\cdot\rho' = \theta(a\epsilon).$$

This equation is due to Helmholtz.

If we remember the Lagrangian variables, it is clear that θ is a function of the initial vector ρ_0 and of t, hence the integral of this equation will take the form

$$a\epsilon = e^{\int\theta dt}\cdot a_0\epsilon_0 = e^{\int -S(\,)\nabla\cdot\rho' dt}a_0\epsilon_0 = \Phi(t)a_0\epsilon_0.$$

But the operator is proved below to be equal to φ itself, so that when $\xi = \nabla u$,

$$a\epsilon = a_0 S\epsilon_0\nabla_0\cdot\rho = +\,a_0\varphi\epsilon_0,$$

or finally we have, if we follow the stream line of a particle, which was implied in the integration above, Cauchy's form of the integral

$$(a/a_0)\epsilon = -\,S\epsilon_0\nabla_0\cdot p,$$

where p is a function of ρ_0 and t. It is evident now if for any particle ϵ is ever zero, that is, $\epsilon_0 = 0$, that always $\epsilon = 0$. This is equivalent to Lagrange's theorem that if for any group of particles of the fluid we have a velocity potential, then that group will always possess a velocity potential. (It is to be noted that velocity potential and vortex are phenomena that belong to the particles and the stream lines, and not to the points of space and the lines of flow.) It must be remembered too, that this result was on the supposition that the density was a function of

the pressure alone, and that the external forces ξ were conservative.

16. We may deduce the equation above as follows, which reproduces in vector form the essential features of Cauchy's demonstration. (Appell, Traité de Méc. III, p. 332.)

Let $d\rho/dt = \sigma$, and $Q = u - \int a dp$, then, remembering that Q is a function of ρ and t, and ρ is a function of ρ_0 and t,

$$d\sigma/dt = \nabla Q(\rho, t).$$

Also $\nabla_0 Q(\rho_0, t) = -\nabla_0 S\rho\nabla Q = -\nabla_0 S\rho d\sigma/dt$, where ∇_0 operates on ρ only; or we can write

$$\nabla_0 Q = \varphi' d\sigma/dt.$$

Hence, operating with $V\nabla_0(\ )$, we have $V\nabla_0\varphi' d\sigma/dt = 0 = d/dt(V\nabla_0\varphi'\sigma)$. Thus the parenthesis equals its initial value, that is, since the initial value of $\varphi'\sigma$ is σ_0, and since $\nabla_0 = \varphi'\nabla$,

$$V\nabla_0\varphi'\sigma = 2\epsilon_0 = V\varphi'\nabla\varphi'\sigma = m_3(\varphi)\varphi^{-1}V\nabla\sigma = 2m_3\varphi^{-1}\epsilon.$$

Thus we have at once $m_3\epsilon = \varphi\epsilon_0$. This is the same as the other form, since $m_3 = a/a_0$. This equation shows the kinematical character of ϵ, and that no forces can set up ϵ or destroy it.

17. The circulation at a given instant of the velocity along any loop is

$$I = -\oint S d\rho\rho'.$$

The time derivative of this is $dI/dt = \oint(\frac{1}{2}Sd\rho\nabla S\rho'\rho' - Sd\rho\rho'') = \oint(-Sd\rho\nabla[\frac{1}{2}\rho'^2 - Q])$. But this is an integral of an exact differential and vanishes. Hence if the forces are conservative and the density depends on the pressure, the circulation around any path does not change as the particles of the path describe their stream lines. The

circulation is an *integral invariant.* This theorem is due to Lagrange. If we express the circulation in the form

$$I = -\int\int Sd\nu\nabla\rho' = -2\int\int Sd\nu\epsilon,$$

we see that the circulation is twice the flux of the vortex through the loop. Hence as the circulation is constant, the flux of the vortex through the surface does not vary in time, if the surface is bounded by the stream loop. The flux of the vortex through any loop at a given instant is the vortex strength of the surface enclosed by the loop. If a closed surface is drawn in the fluid, the flux through it is zero, since the vortex is a solenoidal vector.

18. If we take as our closed surface a space bounded by a vortex tube and two sections of the tube, since the surface integral over the walls of the tube is zero, it follows that the flux of the vortex through one section inwards equals that over the other section outwards. Combining these theorems, it is evident that the vortex strength, or *vorticity*, of a vortex tube is constant. Thus the collection of particles that make up the vortex tube is invariant in time. In a perfect fluid a vortex tube is indestructible, and one could not be generated.

19. It is evident from what precedes that a vortex tube cannot terminate in the fluid but must end either at a wall or a surface of discontinuity, or be a closed tube with or without knots, or it may wind around infinitely in the fluid.

If a vortex tube is taken with infinitesimal cross-section, it is called a vortex filament.

20. We consider next the problem of determining the velocity when the vortex is known. That is, given ϵ, to find $\sigma = \rho'$. We consider first the case of an incompressible fluid, in which the velocity is solenoidal, that is, $S\nabla\sigma = 0$. This with the equations at the boundaries gives us the

following problem: to find σ when $2\epsilon = V\nabla\sigma$, $S\nabla\sigma = 0$, $SU\nu\sigma = 0$ at the boundaries, or if infinite $\sigma_\infty = 0$. This problem has a unique solution, if the containing vessel is simply connected. We cannot enter extensively into it, for it involves the theory of potential functions, and may be reduced to integral equations. However, since $S\nabla\sigma = 0$, we may set $\sigma = V\nabla\tau$, where $S\nabla\tau = 0$, whence

$$\nabla^2\tau = 2\epsilon,$$

and we may suppose τ is known, in the form

$$\tau = \tfrac{1}{2}\pi\cdot\int\int\int\epsilon/T(\rho - \rho_0)dv.$$

If we operate upon this by $V\nabla(\,)$, we find a formula for σ,

$$\sigma = 1/2\pi\cdot\int\int\int V\epsilon(\rho - \rho_0)/T^3(\rho - \rho_0)dv.$$

As we see, this formula is capable of being stated thus: the velocity is connected with its vortex in the same way as a magnetic field is connected with the electric current density that produces it, the vortex filament taking the place of the current, the strength of current being $T\epsilon/2\pi$, and the elements of length of the tube acting like the elements of current. This solution holds throughout the entire fluid, even at points outside the space that is actually in motion with a vortex.

Since the equation of the surface of the tube can be written in the form

$$F(\rho, t) = 0,$$

this surface will move in time. Its velocity of displacement is defined like that of any discontinuity, as $U\nabla F\partial F/\partial t$. On one side the velocity is irrotational, on the other it is vortical. On the irrotational side we have the velocity of the form $\sigma = \nabla P$, and we must have on

that side the same velocity of displacement in the form

$$U\nu SU\nu\nabla P.$$

The energy involved in a vortex on account of the velocity in the particles is

$$\begin{aligned} K &= -\tfrac{1}{2}c\int\int\int\rho'^2 dv \\ &= -\tfrac{1}{2}c\int\int\int S\rho'\nabla\tau dv \\ &= \tfrac{1}{2}c\int\int\int [S\nabla(\rho'\tau) - 2S\tau\epsilon]dv \\ &= \tfrac{1}{2}c\oint\oint Sd\nu\rho'\tau - c\int\int\int S\tau\epsilon dv \\ &= -c\int\int\int S\tau\epsilon dv \quad \text{over all space} \\ &= c/2\pi\cdot\int\int\int\int\int\int S\epsilon\epsilon'/T(\rho-\rho_0)dvdv'. \end{aligned}$$

This is the same formula as that of the energy of two currents. In the expression every filament must be considered with regard to every other filament and itself.

Examples. (1). Let there be first a straight vortex filament terminating at the top and bottom of the fluid. Let all the motion be parallel to the horizontal bottom. Then

$$S\gamma\sigma = 0, \qquad V\gamma\epsilon = 0, \qquad d\epsilon/dt = 0.$$

We have then

$$\sigma = V\gamma\nabla w,\ 2\epsilon = -\gamma\nabla^2 w = 2z\gamma,$$

say,

$$w = -\pi^{-1}\int\int z\log r dA.$$

For a single vortex filament of cross-section dA and strength $k = zdA$, we have

$$\begin{aligned} w &= -k/\pi\log r = -k/\pi\log\sqrt{(x^2+y^2)} \\ \sigma &= V\gamma(\rho-\rho_0)/T^2(\rho-\rho_0)\cdot k/\pi, \end{aligned}$$

where ρ is measured parallel to the bottom.

The velocity is tangent to the circles of motion and inversely as the distance from the vortex filament. The motion is irrotational save at the filament itself.

For the effect of vortices upon each other, and their relative motions, see Webster, Dynamics, p. 518 et seq.

(2). For the case of a vortex ring or a number of vortex rings with the same axis, see Appell, Traité, vol. III, p. 431 et seq.

21. In the more general case in which the fluid is compressible we must resort to the theorem that any vector can be decomposed into a solenoidal part and a lamellar part and these may then be found. The extra term in the electromagnetic analogy would then be due to a permanent distribution of magnetism as well as that arising from the current.

EXERCISES

1. If $S\epsilon\sigma = 0$, then it is necessary and sufficient that $\sigma = M\nabla P$, M being a function of ρ.

2. Discuss the case $V\sigma\epsilon = 0$. Beltrami, Rend. R. Ist. Lomb. (2) 22, fasc. 2.

3. Discuss Clebsch's transformation in which we decompose σ thus, $\sigma = \nabla u + l\nabla V$. Show that the vortex lines are the intersections of the surfaces l and v, and that the lines of flow form with the vortex lines an orthogonal system only when the surfaces l, u, v are triply orthogonal.

4. Discuss the problem of sources and sinks.

5. Consider the problem of multiply-connected surfaces, containing fluids.

22. It will be remembered that Helmholtz's theorem was for the case in which the impressed forces had a potential and the density was a function of the pressure. In this case we will have the equation

$$\partial\sigma/\partial t + 2V\epsilon\sigma = \xi - a\nabla\rho + \tfrac{1}{2}\nabla\sigma^2.$$

Operate by $\frac{1}{2}V\nabla(\ )$ and notice that

$$\partial\epsilon/\partial t - \epsilon S\nabla\sigma - S\sigma\nabla\cdot\epsilon = a^{-1}d(a\epsilon)/dt,$$

whence we have the generalized form

$$a^{-1}d(a\epsilon)/dt + S\epsilon\nabla\cdot\sigma = \tfrac{1}{2}V\nabla\xi - \tfrac{1}{2}V\nabla a\nabla p.$$

If now at the instant t_0 the particle does not rotate and if a is a function of p alone, then at this instant $d\epsilon/dt = \frac{1}{2}V\nabla\xi$, and the particle will acquire an instantaneous increase of its zero vortex equal to the vortex of the impressed force. That is, ξ must be permanently equal to zero if there is to be no rotation at any time.

If $V\nabla\xi = 0$ but a is not a function of p alone, then we have

$$a^{-1}d(a\epsilon)/dt + S\epsilon\nabla\cdot\sigma = -\tfrac{1}{2}V\nabla a\nabla p.$$

The right side is a vector in the direction of the intersection of the isobaric and the isosteric surfaces. Now if we take an infinitesimal length along the vortex tube, l, the cross-section being A, the vorticity is $AT\epsilon = m$, the mass is $cAl = \text{constant} = M$. Then we have, since $a\epsilon = Al\epsilon/M = mlU\epsilon/M$,

$$-S\epsilon\nabla\cdot\sigma = md(lU\epsilon)dtaM = -\frac{T\epsilon}{l}\int U\epsilon\nabla\sigma = \frac{T\epsilon d}{l}\frac{(lU\epsilon)}{dt},$$

$$\begin{aligned} &a^{-1}d(a\epsilon)/dt + S\epsilon\nabla\cdot\sigma = \\ &dm/dt\cdot lU\epsilon/aM + md(lU\epsilon)/a\,Mdt - md(lU\epsilon)/dtaM = \\ &dm/dt\cdot lU\epsilon/aM = V\epsilon\cdot dT\epsilon/dt = \\ &\pm\tfrac{1}{2}\text{ number of tubes.}\end{aligned}$$

Hence the moment m of the vortex will usually change with the time unless the surfaces coincide. Thus a rotating particle may gain or lose in vorticity. If then the isobaric and isosteric surfaces under the influence of heat conditions intersect, vortices will be created along the lines of intersections of the surfaces and these will persist until the surfaces intersect again, save so far as viscosity interferes.

23. Finally we consider the conditions that must be put upon surfaces of discontinuity, in this case of the first order in σ, that is, a wave of acceleration.

Let c be a function of p only. Then

$a\nabla p = dp/dc\, \nabla log\, c$, and the equation of motion becomes

$$\rho'' = \xi - dp/dc \cdot \nabla \log c.$$

Let the equation of the surface of discontinuity be $f(\rho_0, t) = 0$, the normal ν. Let ξ, σ, p, and c be continuous as well as dp/dc, but $\rho'' = \sigma'$ be discontinuous at the surface. Then on the two sides of the surface we have the jump, by p. 263,

$$[\rho''] = - dp/dc[\nabla \log c],$$

or

$$G^2\mu = dp/dc \cdot U\nabla f S\mu U\nabla f.$$

It follows, therefore, that we must have $V\mu U\nabla f = 0$ and $G = \surd(dp/dc)$, or else we have $G = 0$ and $S\mu U\nabla f = 0$. In the first case the discontinuity is longitudinal, in the second transversal. This is Hugoniot's theorem. In full it is:

In a compressible but non-viscous fluid there are possible only two waves of discontinuity of the second order; a longitudinal wave propagated with a velocity equal to $\surd(dp/dc)$, and a transversal wave which is not propagated at all.

The formula for the velocity in the first case is due to Laplace. Also we have for the longitudinal waves $[S\nabla\sigma] = - GS\mu U\nabla f$, for transversal waves equal to zero. On the other hand, for longitudinal waves, $[V\nabla\sigma] = 0$, for transversal, $= GVU\nabla f\mu$.

REFERENCES.

1. Mathematische Schriften (Ed. Gerhart). Berlin, 1850. Bd. II, Abt. 1, p. 20.
2. On a new species of imaginary quantities connected with a theory of quaternions. Proc. Royal Irish Academy, 2 (1843), pp. 424–434.
3. Die lineale Ausdehnungslehre. Leipzig, 1844.
4. Gow: History of Greek Mathematics, p. 78.
5. Ars Magna, Nuremberg, 1545, Chap. 37; Opera 4, Lyon, 1663, p. 286.
6. Algebra. Bologna, 1572, pp. 293–4.
7. Om Directiones analytiske Betejning. Read 1797. Nye Sammlung af det kongelige Danske Videnskabernes Selskabs Skrifter, (2) 5 (1799), pp. 469–518. Trans. 1897. Essai sur la représentation de la direction, Copenhagen.
8. Essai sur une manière de représenter les quantites imaginaires dans les constructions géométriques. Paris, 1806.
9. Theoria residuorum biquadraticum, commentation secunde. 1831.
10. Annales Math. pures et appliquées. 4 (1814–4), p. 231.
11. Theory of algebraic couples, etc. Trans. Royal Irish Acad., 17 (1837), p. 293.
12. Ueber Functionen von Vectorgrossen welche selbst wieder Vectorgrossen sind. Math. Annalen, 43 (1893), pp. 197–215.
13. Grundlagen der Vektor- und Affinor-Analysis. Leipzig, 1914.
14. Lectures on Quaternions. Preface. Dublin, 1853.
15. Note on William R. Hamilton's place in the history of abstract group theory. Bibliotheca Mathematica, (3) 11 (1911), pp. 314–5.
16. Leipzig, 1827.
17. Leipzig.
18. Elements of Vector Analysis (1881–4), New Haven. Vol. 2, Scientific Papers.

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